Lecture 6: Pullbacks and products

<u>Lemma 2</u> Let $Z \hookrightarrow Y \hookrightarrow X$ be regular embeddings. Let

$\pi_{N_Z X}: N_Z X \to Z$, $\pi_{N_Z N_Y X}: N_Z N_Y X \to Z$ be the projections.

Then $\tilde{J}(N_Z X, N_Z N_Y X) \circ \pi^*_{N_Z X} = \pi^*_{N_Z N_Y X}$

Pf: Let ∂, ∂' be the bounding maps associated to

$$\partial$$

$$\begin{tikzcd}
N_Z N_Y X \arrow[r, hook] \arrow[d, "\pi_{N_Z N_Y X}"'] & N_{Z\times\mathbb{A}^1} D(X,Y) \arrow[d, "p"] & N_Z X \times (\mathbb{A}^1 \setminus \{0\}) \arrow[l, hook'] \arrow[d, "\pi_{N_Z X}\times id"] \arrow[dr, "\pi^{N_Z X}"] & \\
Z\times 0 \arrow[r, hook] & Z \times \mathbb{A}^1 & Z \times (\mathbb{A}^1 \setminus \{0\}) \arrow[l, hook'] \arrow[dr, "\pi^Z"'] & N_Z X \arrow[d, "\pi_{N_Z X}"] \\
 & t & & Z
\end{tikzcd}$$

$$\partial'$$

The fact that $Z \hookrightarrow Y \hookrightarrow X$ are regular embeddings implies that $Z\times\mathbb{A}^1 \hookrightarrow D(X,Y)$ is a regular embedding, hence $N_{Z\times\mathbb{A}^1} D(X,Y) \xrightarrow{p} Z\times\mathbb{A}^1$ is a vector bundle over $Z\times\mathbb{A}^1$, in particular p is flat.

As before $\partial' \circ \{t\} \circ \pi^{Z*} = id$ (4.5) and we have

$$\tilde{J}(N_Z X, N_Z N_Y X) \circ \pi^*_{N_Z X} = \partial \circ \{t\} \circ \pi^{N_Z X *} \circ \pi^*_{N_Z X}$$

$$\overset{(4.1)}{=} \partial \circ \{t\} \circ (\pi_{N_Z X} \times \mathrm{id})^* \circ \pi^{Z*}$$

$$\overset{(4.3)}{=} \partial \circ (\pi_{N_Z X} \times \mathrm{id})^* \circ \{t\} \circ \pi^{Z*}$$

$$\overset{(4.4)(2)}{=} \pi^*_{N_Z N_Y X} \circ \partial' \circ \{t\} \circ \pi^{Z*} \overset{(4.5)}{=} \pi^*_{N_Z N_Y X}$$

§ Pullback by a regular embedding

<u>Def</u> Let $i : Y \hookrightarrow X$ be a regular embedding in Sch_k. Let $\pi_{N_Y X} : N_Y X \to Y$ be the projection and let $r^X_Y : N_Y X \to Y$ be the homotopy inverse to $\pi^*_{N_Y X}$ given by §?

Define $i^! : X \to Y$ by

$$i^! := r^X_Y \circ \tilde{J}(X, Y)$$

<u>Note</u> r^X_Y depends on a choice of coordination for $N_Y X \to Y$, we ignore this, and also assume all coordinations are suitably compatible, as needed.

<u>Theorem</u>: Let $Z \xrightarrow{j} Y \xrightarrow{i} X$ be regular embeddings. Then $(ij)^!$ is homotopic to $j^! i^!$.

We first prove

Lemma 3 let
$$\begin{tikzcd} Y' \arrow[r, hook, "i'"] \arrow[d, "f'"'] & X' \arrow[d, "f"] \\ Y \arrow[r, hook, "i"'] & X \end{tikzcd}$$
be a cartesian diagram with i a regular embedding and f flat.

Then $f'^{*} i^{!} \sim i'^{!} f^{*}$

Proof By (9.5) the homotopy inverse (11.3) are compatible with flat pullback, i.e. from the cartesian square

$$\begin{tikzcd} Y' \times_Y N_Y X = N_{Y'}X' \arrow[r, shift left, "r^{X'}_{Y'}"] \arrow[r, shift right, "\pi'"'] \arrow[d, "N(f')"'] \arrow[dr, phantom, "\ulcorner"] & Y' \arrow[d, "f'"] \\ N_Y X \arrow[r, shift left, "\pi"] \arrow[r, shift right, "r^X_Y"'] & Y \end{tikzcd}$$

we have

$$(**) \quad f'^{*} r^X_Y \sim r^{X'}_{Y'} N(f')^{*}$$

By (11.3) we have
$$J(X',Y') \circ f^{*} = N(f')^{*} \circ J(X,Y)$$

$$\Rightarrow \quad r^{X'}_{Y'} \circ J(X',Y') \circ f^{*} = r^{X'}_{Y'} \circ N(f')^{*} \circ J(X,Y)$$
$$\overset{(**)}{\sim} f'^{*} r^X_Y \circ J(X,Y)$$

$$\Rightarrow \quad i'^{!} f^{*} \sim f'^{*} i^{!} \qquad \square$$

Proof of Theorem : We have the diagram (s_Y = 0-section)

$$(\mathrm{I})\qquad \begin{tikzcd} N_Y X \arrow[r, shift left, "\pi_{N_Y X}"] \arrow[dr, leftrightarrow, "s_Y \circ j"'] & Y \arrow[l, shift left, "s_Y"] \arrow[d, "i"] \\ & Z \end{tikzcd}$$

with $\pi^{N_Y X}$ smooth.

This induces the diagram of vector bundles

$$\begin{tikzcd} N_Z N_Y X \arrow[r, "\pi^Z_{N_Z Y}"] \arrow[dr, "\pi_{N_Z N_Y X}"'] & N_Z Y \arrow[d, "\pi_{N_Z Y}"] \\ & Z \end{tikzcd}$$

with $\pi^Z_{N_Z Y} = N(\pi_{N_Y X})$.

By lemma (11.5) applied to diagram (I) we have

$$(*)\qquad J(N_Y X, Z) \circ \pi^*_{N_Y X} = \pi^{Z\,*}_{N_Z Y} \circ J(Y,Z)$$

We have the corresponding strong homotopy inverses

$$\begin{tikzcd} N_Z N_Y X \arrow[r, "r^X_{Y,Z}"] \arrow[dr, "r^{N_Y X}_Z"'] & N_Z Y \arrow[d, "r^Y_Z"] \\ & Z \end{tikzcd}$$

to $\pi^*_{N_Z Y}$, $\pi^{Z\,*}_{N_Z Y}$, $\pi^*_{N_Z N_Y X}$

One shows using the standard composition of homotopies:

(**) $r^Y_Z\, r^X_{Y,Z}$ is a strong homotopy inverse to $\pi^*_{N_Z N_Y X}$ and is homotopic to $r^{N_Y X}_Z$.

We have homotopies

$$\bar{J}(N_Y X, Z)\circ \bar{J}(X,Y) \overset{\text{(Lemma 1)}}{\sim} \tilde{J}(N_Z X, N_Z N_Y X)\circ \bar{J}(X,Z)$$

(***)

$$\wr \longleftarrow (\pi^* \rho \sim \mathrm{id}) \longrightarrow \wr$$

$$\bar{J}(N_Y X, Z)\,\pi^*_{N_Y X}\circ \rho^X_Y\, \bar{J}(X,Y) \qquad \tilde{J}(N_Z X, N_Z N_Y X)\circ \pi^*_{N_Z X}\circ \rho^X_Z\, \bar{J}(X,Z)$$

Thus:

$$j^! \circ i^! = \rho^Y_Z\, \bar{J}(Y,Z)\circ \rho^X_Y\, \bar{J}(X,Y)$$

$$\sim \rho^Y_Z\, \rho^X_{Y,Z}\, \pi^{Z\,*}_{N_Z Y}\, \bar{J}(Y,Z)\circ \rho^X_Y\, \bar{J}(X,Y)$$

$$\overset{(*)}{=} \rho^Y_Z\, \bar{J}(N_Y X, Z)\circ \pi^*_{N_Y X}\circ \rho^X_Y\, \bar{J}(X,Y)$$

$$\overset{(**)}{\sim} \rho^{N_Y X}_Z\, \bar{J}(N_Y X, Z)\circ \pi^*_{N_Y X}\circ \rho^X_Y\, \bar{J}(X,Y)$$

$$\overset{(***)}{\sim} \rho^{N_Y X}_Z\, \tilde{J}(N_Z X, N_Z N_Y X)\, \tilde{\pi}^*_{N_Z X}\circ \rho^X_Z\, \bar{J}(X,Z)$$

$$\overset{\text{(Lemma 2)}}{=} \rho^{N_Y X}_Z\, \bar{\pi}^*_{N_Z N_Y X}\circ \rho^X_Z\, \bar{J}(X,Z)$$

$$= \rho^X_Z\, \bar{J}(X,Z) = (ij)^!$$

($\rho^{N_Y X}_Z$ is a strong homotopy inverse to $\pi^*_{N_Z N_Y X}$)

□

§ Pull backs for morphisms with smooth target

Def Let $f: Y \to X$ be a morphism in Sch_k with X smooth over k. Factor f as

$$\begin{tikzcd} Y \arrow[r, hook, "\gamma_f"] \arrow[dr, bend right, "f"'] & Y \times X \arrow[d, "p_2"] \\ & X \end{tikzcd}$$

where γ_f is the graph embedding $\gamma_f = (id, f)$. Define

$$\tilde{f}^* = \gamma_f^! \, p_2^*$$

(The notation $\tilde{f}^*$ is temporary, we will write this as f^* after showing all the pullbacks agree, see the next proposition.)

Note γ_f is a regular embedding with normal bundle $f^* T_X$, and p_2 is flat since $Y \to \operatorname{Spec} k$ is flat.

Prop (1) If $Z \xrightarrow{g} Y \xrightarrow{f} X$ are morphisms in Sch_k with
(Thm 12.1) $X, Y \in Sm_k$, then $\widetilde{(fg)}^* \simeq \tilde{g}^* \tilde{f}^*$

(Prop 12.2) 2) If $f: Y \to X$ is flat then $\tilde{f}^* = f^*$

(Prop 12.3) 3) If $i: Y \hookrightarrow X$ is closed immersion in Sm_k, then i is a regular embedding and $i^! \simeq \tilde{i}^*$

We first prove

Lemma 3 Let
$$\begin{tikzcd} Y' \arrow[r, hook, "i'"] \arrow[d, "f'"'] & X' \arrow[d, "f"] \\ Y \arrow[r, hook, "i"'] & X \end{tikzcd}$$
be a cartesian diagram with i a regular embedding and f flat.

Then $f'^{*} i^{!} \sim i'^{!} f^{*}$

Proof By (9.5) the homotopy inverses are (11.3) compatible with flat pullbacks, i.e. from the cartesian square

$$\begin{tikzcd} Y' \times_Y N_Y X = N_{Y'}X' \arrow[r, "\pi'"] \arrow[r, phantom, bend left, "r^{X'}_{Y'}"] \arrow[d, "N(f')"'] & Y' \arrow[d, "f'"] \\ N_Y X \arrow[r, "\pi"'] \arrow[r, phantom, bend right, "r^{X}_{Y}"'] & Y \end{tikzcd}$$

we have

$$(**) \quad f'^{*} r^{X}_{Y} \sim r^{X'}_{Y'} N(f')^{*}$$

By (11.3) we have
$$J(X',Y') \circ f^{*} = N(f')^{*} \circ J(X,Y)$$

$$\Rightarrow \quad r^{X'}_{Y'} \circ J(X',Y') \circ f^{*} = r^{X'}_{Y'} \circ N(f')^{*} \circ J(X,Y)$$
$$\overset{(**)}{\sim} f'^{*} r^{X}_{Y} \circ J(X,Y)$$

$$\Rightarrow \quad i'^{!} f^{*} \sim f'^{*} i^{!} \qquad \square$$

Proof (1)

We have the commutative diagram, the square is cartesian

dh

$$\begin{tikzcd}
Z \arrow[r, hook, "\gamma_g"'] \arrow[rrr, bend left, "\gamma_{fg}"] \arrow[dr, "g"'] & Z\times Y \arrow[r, hook, "id_Z\times\gamma_f"'] \arrow[d, "p_Y"] & Z\times Y\times X \arrow[r, "p_{Z,X}"] \arrow[d, "p_{Y\times X}"] \arrow[dd, bend left, "g"] & Z\times X \arrow[ddl, bend left, "p_X^{Z\times X}"] \\
& Y \arrow[r, hook, "\gamma_f"'] \arrow[dr, "f"'] & Y\times X \arrow[d, "p_X^{Y\times X}"] & \\
& & X &
\end{tikzcd}$$

Let $i = (id_Z\times\gamma_f)\circ\gamma_g$, note that $p_{Z,X}$ is smooth, since Y is smooth/k. Then:

$$(fg)^{\tilde{*}} = \gamma_{fg}^{!}\circ p_X^{Z\times X\,*} = i^{!}\circ p_{Z,X}^{*}\, p_X \qquad (11.5)$$

$$\simeq \gamma_g^{!}\,(id_Z\times\gamma_f)^{!}\, p_{Y\times X}^{*}\, p_X^{Y\times X\,*} \qquad \text{(Theorem)}$$

$$\simeq \gamma_g^{!}\, p_Y^{*}\, \gamma_f^{!}\, p_X^{Y\times X\,*} \qquad \text{(Lemma 3)}$$

$$= g^{\tilde{*}}\, f^{\tilde{*}}$$

(2) $f^{\tilde{*}} = \gamma_f^{!}\circ p_X^{*} = r_{\gamma_f}\circ J(\gamma_f)\circ p_X^{*}$

We have

$$\begin{tikzcd}
Y \arrow[r, hook, "\gamma_f"] \arrow[dr, "f"'] & Y\times X \arrow[d, "p_X^{Y\times X}"] \\
& X
\end{tikzcd}$$

. Note that for

$$N_{\gamma_f} = f^* T_X \quad \underbrace{X \xrightarrow{\pi} Y \xrightarrow{f} X}_{g}$$

π is flat of relative dimension $\dim X$, f is flat of relative dimension $\dim Y - \dim X$,

so g is flat of relative dimension $\dim Y$. By (11.4)

we have $\quad J(\gamma_f) \circ p_X^* = g^*$

so $\quad \tilde{f^*} = r_Y^{Y \times X} g^* = r_{f^* T_X} \circ \pi^* \circ f^* = f^*$

13) A closed immersion in Sm_k is a regular imbedding

$$\begin{tikzcd} Y \arrow[r, hook, "\gamma_i"] \arrow[dr, "i"'] & Y \times X \arrow[d, "p_X"] \\ & X \end{tikzcd}$$

with p_X smooth, giving the surjective map of normal bundles $N(p_X) : N(\gamma_i) \to N(i)$

By (11.5) we have $\quad J(\gamma_i) \circ p_X^* = N(p_X)^* \circ J(i)$

$$i^{\tilde{*}} = \gamma_i^! \circ p_X^* = r_Y^{Y \times X} J(\gamma_i) p_X^* = r_Y^{Y \times X} N(p_X)^* \circ J(i) = r_Y^X \circ J(i) = i^! \quad \square$$

<u>Notation</u> Write f^* for $\tilde{f^*}$ and for $i : Y \hookrightarrow X$ a closed immersion in Sm_k, write i^* for $i^!$.

<u>Lemma</u> 5 Let
$$\begin{tikzcd} Y' \arrow[r, hook, "i'"] \arrow[d, "g'"'] & X' \arrow[d, "g"] \\ Y \arrow[r, hook, "i"'] & X \end{tikzcd}$$
be cartesian with g flat and proper.

Then g, g' induces $N(g) : N_{Y'}X' = Y' \times_Y N_Y X \to N_Y X$ flat and proper, and we have

$$J(X,Y) \circ g_* \simeq N(g)_* \circ J(Y',X')$$

Pf We have

$$\begin{tikzcd} N_{Y'}X' \arrow[r, hook] \arrow[d, "N(g)"'] \arrow[rr, bend left, "\partial'"] & D(X',Y') \arrow[d, "D(g)"] & X' \times (\mathbb{A}^1 - 0) \arrow[l, hook'] \arrow[d, "g \times id"] \arrow[r, "\pi_{X'}"] & X' \arrow[d, "g"] \\ N_Y X \arrow[r, hook] \arrow[rr, bend right, "\partial"'] & D(X,Y) & X \times (\mathbb{A}^1 \setminus \{0\}) \arrow[l, hook'] \arrow[r, "\pi_X"'] & X \end{tikzcd}$$

$D(g)$ flat

$$J(X,Y) \circ g_* \simeq \partial \circ \{t\} \circ \pi_X^* \circ g_* \simeq \partial \circ \{t\} \circ (g \times id)_* \circ \pi_{X'}^*$$
$$\simeq \partial \circ (g \times id)_* \circ \{t\} \circ \pi_{X'}^*$$
$$\simeq N(g)_* \circ \partial' \circ \{t\} \circ \pi_{X'}^*$$
$$\simeq N(g)_* \circ J(X',Y')$$
$\square$

Prop (13.5) Let
$$\begin{tikzcd} Y' \arrow[r, "g'"] \arrow[d, "f'"'] & Y \arrow[d, "f"] \\ X' \arrow[r, "g"'] & X \end{tikzcd}$$
be a cartesian square in $Sch/_k$ with g smooth and proper & X (hence X') smooth$/_k$.

Then $f^* g_* \simeq g'_* f'^*$

[This extends to the case of a Tor-independent cartesian square with X, X' in Sm_k; g proper]

Pf Consider the diagram

$$\begin{tikzcd} Y' \arrow[r, hook] \arrow[d, equal] & Y' \times X' \arrow[r, "p^{Y'}_{X'}"] \arrow[d, "g' \times id"] & X' \arrow[d, equal] \\ Y' \arrow[r, hook, "{(g,f')}"'] \arrow[d, "g'"'] & Y \times X' \arrow[r, "p^{Y}_{X'}"] \arrow[d, "id \times g"] & X' \arrow[d, "g"] \\ Y \arrow[r, hook, "\gamma_f"'] & Y \times X \arrow[r, "p^Y_X"] & X \end{tikzcd}$$

$Y' = Y \times_X X'$

$Y \hookrightarrow Y \times X$

$Y \times_k X' \hookrightarrow Y \times X \times_k X'$

$N_{Y', Y \times X'} = (fg')^* T_X$

$N^Y_X \to Y'$

We have 3 maps $X' \to Y$

$g'_* \circ p^{Y'}_{X'} \circ J(\gamma_{f'}) \circ p^{Y'*}_{X'}$

$g'_* \circ p^{Y}_{X} \circ J((g,f')) \circ p^{Y*}_{X'}$

$p^{Y}_{X} \circ J(\gamma_f) \circ p^{Y*}_{X} \circ g_*$

By (11.5) we have $r^{y'}_{x'} \bar{J}(\gamma_{f'}) \sim r^{y'}_{x} \circ \bar{J}((g,f))$

and by lemma 5 we have

$$r^{y}_{x} \bar{J}(\gamma_f) \circ (id \times g)_* = r^{y}_{x} \circ N(g)_* \bar{J}((g,f))$$
$$\sim g'_* r^{y'}_{x} \bar{J}((g,f))$$

so

$$g'_* \circ f'^* \sim g'_* \circ r^{y'}_{x} \bar{J}((g,f)) \circ p^{y_*}_{x'}$$
$$\sim r^{y}_{x} \bar{J}(\gamma_f) (id \times g)_* \circ p^{Y_*}_{X'}$$
$$= r^{y}_{x} \bar{J}(\gamma_f) p^{Y_*}_{X} \circ g_*$$
$$= f^* g_* \qquad \square$$

Products We assume $\mathcal{M}$ has associative products $\cdot : \mathcal{M} \times \mathcal{M} \to \mathcal{M}$ compatible with $\mathbb{R}^n_0$ module structure, ρ_* for $\rho : F \hookrightarrow L$ projective families, ρ^*: $\rho : F \hookrightarrow L$ finite and

$$\partial_v(\alpha \cdot \beta) = \partial_v(\alpha) \cdot s^{\pi}_v(\beta) + (-1)^n s^{\pi}_v(\alpha) \partial_v(\beta) + (-1) \cdot \partial_v(\alpha) \partial_v(\beta)$$

Def Suppose M has products, then for $X, Y \in Sch_k$ we have (cross product)

$$\boxtimes : C_\bullet(X, M) \otimes C_\bullet(Y, M) \to C_\bullet(X \times Y, M)$$

induced by: take $x \in X_{(p)}$, $y \in Y_{(q)}$

let $z_1 \in (X \times Y)_{(p+q)}$, $\vdots$, z_r be the generic points of $\overline{\{x\}} \times \overline{\{y\}} \subset X \times Y$

$z_i \xrightarrow{p_{i1}} x$, $z_i \xrightarrow{p_{i2}} y$. Then define

$$\boxtimes_{(x,y)} : M(k(x)) \times M(k(y)) \to \bigoplus_i M(k(z_i))$$

$$(\alpha, \beta) \mapsto \bigoplus_i (p_{i,1}^*)_*(\alpha) \cdot (p_{i,2}^*)_*(\beta)$$

One shows $\sum_{x,y} \boxtimes_{(x,y)}$ defines a map of complexes

(possibly replacing differential with $-$differential)

Def for $X \in Sm_k$, let $\Delta_X : X \hookrightarrow X \times X$ be the diagonal and define

$$\cup : C^\bullet(X, M) \times C^\bullet(X, M) \to C^\bullet(X, M)$$

by $\alpha\cdot\beta = \Delta_X^*(\alpha\boxtimes\beta)$

This gives $\bigoplus A^p(X,M,q)$ the structure of a bigraded associative ring

$$\cup_{(p,q),(p',q')} : A^p(X,M,q)\times A^{p'}(X,M,q') \to A^{p+p'}(X,M,q+q')$$

We just list the properties without prove

(a) $\boxtimes$ and $\cup$ are associative

(b) for $\tau : X\times Y \to Y\times X$ $\quad \tau^* = (-1)^{(q-p)(q'-p')}$ on $C_p(X,M,q)\times C_{p'}(X,M,q')$

and $\cup$ is similarly graded commutative.

For

$a\in K_n^M(k)$, $b\in K_m^M(k)$, $\alpha\in C_p(X,M,q)$, $\beta\in C_{p'}(Y,M,q')$

we have $a\cdot\alpha \underset{X,Y}{\boxtimes} b\cdot\beta = (-1)^{m(q-p)} ab\cdot(\alpha\underset{X,Y}{\boxtimes}\beta)$ & similarly for $\underset{X}{\cup}$

(c) $f^*\boxtimes g^* = (f\times g)^*$ $\quad f_*\boxtimes g_* = (f\boxtimes g)_*$

for f,g with smooth target

or f,g flat or f,g regular embedding $\quad$ for f,g proper

(d) for $f: Y \to X$ proper X smooth we have $\begin{tikzcd} Y \arrow[r, "(id, f)"] \arrow[d, "f"] & Y \times X \arrow[d, "f \times id"] \\ X \arrow[r, "\Delta"] & X \times X \end{tikzcd}$

$$f^{X}(f_{*}\alpha \cup \beta) = \alpha \cup f_{*}(\beta)$$ (θ) + (12.5) extended

(e) $\mathring{A}(X, M)$ is an $\mathring{A}(X, K^M)$ - module with properties corresp to (a)-(d)

<u>Examples</u> X smooth

1. $CH^p(X) = A^p(X, K^M, p)$ and $\cup$ is the standard ring structure on $CH^*(X)$ following Fulton

2. $\alpha_1, \alpha_2 \in k^\times$ D_1, D_2 divisors on X intersecting properly $D_1 \cdot D_2 = \sum_i m_i Z_i \in C^2(X, 2)$

so $(1 \otimes k(d_1)) \cup_X (1 \otimes k(d_2)) = \sum_i m_i \otimes k(z_i)$

$m_i = \lg \mathcal{O}_{X,Z_i} / (g_1 g_2) \mathcal{O}_{X,Z_i}$ g_i local defining gen for D_i

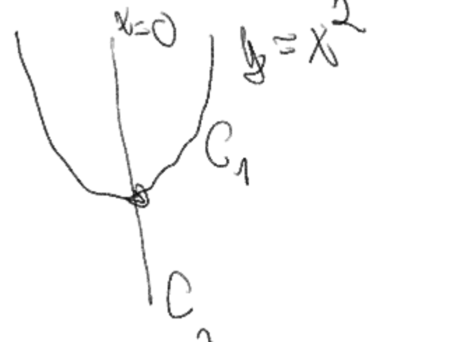
d_1, d_2, z_i gen pts of D_1, D_2, Z_i. Then using the $K^M_*(k)$ algebra structure on $C^*(X, *)$ we have

$$(\{\alpha_1\} \otimes k(d_1)) \cup_X (\{\alpha_2\} \otimes k(d_2)) = [\{\alpha_1\} \cdot (1 \otimes k(d_1))] \cup_X [\{\alpha_2\} \cdot (1 \otimes k(d_2))]$$

$$= (\{\alpha_1\} \cdot \{\alpha_2\}) \cdot [(1 \otimes k(d_1)) \cup_X (1 \otimes k(d_2))] = \{\alpha_1, \alpha_2\} \cdot \sum_i m_i \otimes k(z_i)$$

$$= \sum_i \{\alpha_1, \alpha_2\}^{m_i} \otimes k(z_i)$$

3. Take $X = \mathbb{A}^2 = \operatorname{Spec} k[x,y] \subset \bar{X} = \mathbb{P}^2$



$C_1 = V(y-x^2)$, $C_2 = V(x) \subset \mathbb{A}^2$, c_i gen pt of C_i

$Y = V(y) \subset \mathbb{A}^2$ η gen pt of Y

Let $\alpha = \{x\}\otimes k(c_1) - \{y\}\otimes k(c_2) \in C^1(X,2)$

$\beta = 1 \otimes k(\eta) \in C^1(X,1)$

one checks $d\alpha = 0$, $d\beta = 0$

$\beta = i_{Y*}(1\otimes k(\eta))$

$\bar{Y}$ = closure of Y in $\mathbb{P}^2$, then $\beta = i_{\bar{Y}*}(1\otimes k(\eta))$

$$1\otimes k(\eta) = 1 \in C^0(\bar{Y},0)$$

so

$$\alpha \cup_X \beta = \alpha \cup_{\bar{X}} i_{\bar{Y}*}(1\otimes k(\eta))$$

$$= i_{\bar{Y}*}(i^*_{\bar{Y}}(\alpha) \cup_Y 1) \overset{!}{=} i_{\bar{Y}*}(i^*_{\bar{Y}}(\alpha))$$

so we compute $i^*_{\bar{Y}}(\alpha) \sim i^*_Y(\alpha)$, and then push forward

Since $\bar{Y} \cap (\bar{C}_1 \cup \bar{C}_2) \subset \mathbb{A}^2$ we restrict to $\mathbb{A}^2 = X$ & compute i^*_Y

$D(X,Y) = \operatorname{Spec}_{k[x,y,t]} k[x,y,t,\tfrac{y}{t}] = \operatorname{Spec}_{k[x,y,t]} k[x,z,t]$ via $y \mapsto tz$ (Rost convention)

$N_Y X = V(t) = \operatorname{Spec} k[x,z] = Y \times \mathbb{A}^1 \to Y$

$$Y \hookleftarrow N_Y X \hookrightarrow D(X,Y) \hookleftarrow X \times \mathbb{A}^1 \setminus \{0\} \xrightarrow{\pi} X = \operatorname{Spec} k[x,y]$$

$Y = \operatorname{Spec} k[x]$, $N_Y X = \operatorname{Spec} k[x,z]$, $D(X,Y) = \operatorname{Spec} k[x,z,t] \to \operatorname{Spec} k[x,y]$, $X \times \mathbb{A}^1\setminus\{0\} = \operatorname{Spec} k[x,z,t,t^{-1}] = \operatorname{Spec} k[x,y,t,t^{-1}]$ $(z = y t^{-1})$

Compute

$J(X,\mathbb{D})(\{x\}\otimes k(c_1)) = \partial(\{t,x\}\otimes k(c_1,t))$;

$(c_1,t) = $ gen pt of $V(y-x^2) \subset X \times \mathbb{A}^1 \setminus \{0\} = V(tz+x^2) \cap X \times \mathbb{A}^1 \setminus \{0\}$

$\qquad\qquad\qquad\qquad\qquad \overset{\shortparallel}{V(z+t^{-1}x^2)}$

$[N_Y X \cap V(tz+x^2)] = V(x^2)$, so $\partial \leadsto \partial_x$ for $V(x,t) \subset V(x^2+tz)$

Note $V(x^2+tz) \setminus V(z) \subset \operatorname{Spec} k[x,z,t]$ is $V(\frac{x^2}{z}-t) \subset \operatorname{Spec} k[x,t,z,z^{-1}]$ Spec $k[z]$

$$V(x^2+tz) \setminus V(z) = V\left(\frac{x^2}{z}-t\right) \subset \operatorname{Spec} k[x,t,z,z^{-1}]$$

$$\cong$$

$$\operatorname{Spec} k[x,z,z^{-1}] \text{ with } t = a\cdot x^2$$

so $\partial = \partial_x$ on $k(z)[x]_{(x)}$ with $t = \frac{-1}{z}x^2$, $k(c_1,t) \cong k(x,z)$

Thus

$$\partial(\{t,x\}\otimes k(c_1,t)) = \partial_x\left(\left\{\frac{-x^2}{z},x\right\}\otimes k(x,z)\right)$$

$$= \left\{(-1)^{\operatorname{ord}(x^2/z)\operatorname{ord}(x)} \cdot \frac{x^{\operatorname{ord}(x^2/z)}}{(-x^2/z)^{\operatorname{ord}(x)}}\right\}\otimes k(e) = \{-z\}\otimes k(e)$$

Next

$J(X,y)(\{y\}\otimes k(c_2)) = \partial(\{t,y\}\otimes k(c_2,t))$

$(c_2,t) = $ gen pt of $V(x) \subset X \times \mathbb{A}^1 \setminus \{0\} = \operatorname{Spec} k[x,z,t]/x \cap X \times (\mathbb{A}^1 \setminus \{0\})$

so $\partial = \partial_t \leadsto (t) \subset k(z)[t]$ and $\qquad \overset{\shortparallel}{\operatorname{Spec} k[z,t,t^{-1}]}$

$\partial(\{t,y\}\otimes k(c_2,t)) = \partial_t(\{t,-zt\}\otimes k(z,t))$

$$= \{(-1)^{\mathrm{ord}(x)\,\mathrm{ord}(-zt)} \frac{(-zt)^{\mathrm{ord}(t)}}{t^{\mathrm{ord}(-zt)}}\} \otimes k(0) = \{z\} \otimes k(0)$$

$$\Rightarrow J(X,Y)(\alpha) = \{-1\} \otimes k(0) = \pi^*_{N_y X}(\{-1\} \otimes k(0,0))$$

$$\Rightarrow i^!_y(\alpha) = r^y_X J(X,Y)(\alpha) = r^y_X \pi^*_{N_y X}(\{-1\} \otimes k(0,0)) = \{-1\} \otimes k(0,0)$$

Thus

$$\alpha \cup \beta = i_{y*}(\{-1\} \otimes k(0,0)) = \{-1\} \otimes k(0,0) \in C^2(\mathbb{A}^2, K^M, 3)$$

and in $A^*(\mathbb{P}^2, *)$ we have

$$\tilde{\alpha} = [x \otimes k(C_1) - y \otimes k(C_2)] \in A^1(X, 2) \quad \text{and}$$

$$\bar{\beta} = [1 \otimes k(y)] \in A^1(X, 1)$$

$$\tilde{\alpha} \cup \bar{\beta} = [\{-1\} \otimes k(0,0)] \in A^2(\mathbb{P}^2, 3) \cong K^M_1(k)$$

$$\uparrow$$

$$i_{(0,0)*}$$

We can check this in $A^*(\mathbb{P}^2, *)$

Let $\gamma = \{x, y - x^2\} \otimes k(\tau)$, τ = generic point of $\mathbb{P}^2$, use homog. coords X, Y, Z on $\mathbb{P}^2$

$$\partial\gamma = \partial_{(X=0)}\gamma + \partial_{(YZ=X^2)}\gamma + \partial_{(Z=0)}\gamma$$

$$= \{y - x^2\}|_{(x=0)} \otimes k(C_2) - \{x\}|_{y=x^2} \otimes k(C_1) + \partial_{(Z=0)}\gamma$$

$$= \{y\} \otimes k(C_2) - \{x\} \otimes k(C_1) + \partial_{(Z=0)}\gamma = \partial_{(Z=0)}\gamma - \alpha$$

For $\partial_{(Z=0)}\gamma$ we pass to the affine $Y \neq 0$ with coords

$x' = X/Y$, $z = Z/Y$. Since $x = X/Z$ we have $x'/z = x$

and $y = Y/Z = 1/z$.

$$\text{so } \partial_{(Z=0)}\gamma = \partial_{(Z=0)}\left(\{x, y-x^2\}\otimes k(\gamma)\right)$$

$$= \partial_{(Z=0)}\left(\left\{\frac{x'}{z}, \frac{z-x'^2}{z^2}\right\}\otimes k(\gamma)\right) \qquad C_3 = \text{generic points of } (Z=0)$$

$$= \left\{(-1)^{2\cdot 1}\cdot \left(z-x'^2/z^2\right)^{-1} \Big/ \left(\frac{x'}{z}\right)^{-2} \Big|_{z=0}\right\}\otimes k(C_3)$$

$$= \left\{x'^2/(z-x'^2)\Big|_{z=0}\right\}\otimes k(C_3)$$

$$= \{-1\}\otimes k(C_3)$$

So $[\alpha] = [\{-1\}\otimes k(C_3)]$ in $A^1(\mathbb{P}^2, 2)$

We can further modify to

$$\{-1\}\otimes k(C_3) + \partial\left(\{X/Z, -1\}\otimes k(\gamma)\right)$$

$$= \{-1\}\otimes k(C_3) + \partial_{(X=0)}(\{x,-1\}\otimes k(\gamma)) + \partial_{Z=0}(\{1/z, -1\}\otimes k(\gamma))$$

$$= \{-1\}\otimes k(C_3) + \{-1\}\otimes k(C_2) - \{-1\}\otimes k(C_3)$$

$$= \{-1\}\otimes k(C_2) \implies [\alpha] = [\{-1\}\otimes k(C_2)] \text{ in } A^1(\mathbb{P}^2, 2)$$

$$\implies [\alpha]\cup[\beta] = [\{-1\}\text{ on }(Y\text{-axis})]\cup[X\text{-axis}] = [\{-1\}\text{ on }(0,0)] \quad \checkmark$$

§ Pull back for lci morphisms

<u>Def</u> 1) An lci morphism $f: Y \to X$ is a morphism in Sch_k that factors as

$$\begin{tikzcd} Y \arrow[r, hook, "i"] \arrow[dr, "f"'] & P \arrow[d, "g"] \\ & X \end{tikzcd}$$

with i a regular embedding and p smooth.

2) Let $f: Y \to X$ be an lci morphism factored as in (1). Define $f^!: X \rightsquigarrow Y$ by $f^* = i^! g^*$

<u>Proposition</u> 1) for $f: Y \to X$ an lci morphism, f^* is independent of the choice of factorization $f = gi$, up to homotopy.

2) If $Z \xrightarrow{g} Y \xrightarrow{f} X$ is a sequence of lci morphisms then $fg: Z \to X$ is also lci, and $(fg)^* = g^* f^*$

Pf (1) Given

$$\begin{tikzcd} & P_1 \arrow[dr, "g_1"] & \\ Y \arrow[ur, hook, "i_1"] \arrow[dr, "i_2"'] & & X \\ & P_2 \arrow[ur, "g_2"'] & \end{tikzcd}$$

we have

$$\begin{tikzcd} Y \arrow[r, hook, "i=(i_1,i_2)"] \arrow[dr, "i_1"'] & P_1 \times_X P_2 \arrow[d, "p_1"] \arrow[dr, "g"] & \\ & P_1 \arrow[r, "g_1"'] & X \end{tikzcd}$$

Then i is a regular embedding and p_1 is smooth

By (11.5) we have $J(i)\circ p_1^* = N(p_1)^* \circ J(i_1)$
with $N(p_1) : N_Y(P_1 \times_X P_2) \to N_Y P_1$ induced from p_1

Then $i^! g^* = r_i J(i) p_1^* g_1^*$

$$= r_i N(p_1)^* J(i_1) g_1^*$$

$$\sim r_{i_1} J(i_1) g_1^* = i_1^! g_1^*$$

and similarly $i^! g^* \sim i_2^! g_2^*$.

(2) Given

$$\begin{tikzcd}
Z \arrow[r, hook, "i_2"'] \arrow[dr, "g"'] & P_2 \arrow[d, "q_2"] & \\
& Y \arrow[r, hook, "i_1"'] \arrow[dr, "f"'] & P_1 \arrow[d, "q_1"] \\
& & X
\end{tikzcd}$$

We have the commutative diagram

$i_2' = (g, i_2)$, $Y \times_X P_2 = Y \times_{P_1} P_1 \times_X P_2$

$$\begin{tikzcd}
Z \arrow[rr, hook, "i_2'"] \arrow[dr, "i_2"'] \arrow[ddrr, bend right, "g"'] & & Y \times_{P_1} P_1 \times_X P_2 \arrow[r, hook, "i_1'"] \arrow[dl] \arrow[dd, "p_1'"] & P_1 \times_X P_2 \arrow[dd, "p_1"] \\
& P_2 \arrow[dr, "q_2"] & & \\
& & Y \arrow[r, hook, "i_2"] \arrow[dr, "f"'] & P_1 \arrow[d, "q_2"] \\
& & & X
\end{tikzcd}$$

with i_2', i_1' regular embeddings, p_1', p_1 smooth. Thus fg is lci and

$$(fg)^! \sim (i_1' i_2')^! \circ (p_2 p_1)^*$$

$$\sim i_2'^! \, i_1'^! \circ p_1^* \, p_2^*$$

$$\sim i_2'^! \, p_1'^* \circ i_2^! \, p_2^! \qquad \text{(Lemma 4)}$$

$$\sim g^! \circ f^! \qquad \square$$