Milnor-Witt K-theory and the Rost-Schmidt complex

Motivos Seminar - SS 2025

"Quadratic Intersection and motivic linking"

Lecture 8, May 28th

Plan for today:

1.- Present $GW(F)$, $W(F)$

2.- Introduce $K_*^{MW}(F)$ and connections with.

K_*^M, $GW(F)$, $W(F)$

3.- Residue maps.

- non-canonical
- canonical
- twisted

need of twisted MW-K-theory

4.- Rost-Schmid complex

5.- The 5 básic maps.

Main References:

[Lem 23] Lemariè.- Motivic Knot theory

[Mor 12] Morel - $\mathbb{A}^1$-alg. topology / F

[Fas 20] Lectures on Chow-Witt gps.

Additional References:

[Mor 03] Morel: An introduction to $\mathbb{A}^1$-homotopy theory

[Feld 21] MW-sheaves and modules

§ 1. Grothendieck - Witt ring and the Witt ring

Symmetric bilinear forms / F

Let F be a field, V a finite dim vector space / F

A symmetric bilinear form on V is a bilinear map

$$b : V \times V \longrightarrow F$$

such that $b(u, v) = b(v, u)$

b is non-degenerated if for all $u \in V$

$$V \longrightarrow V^* = \mathrm{Hom}(V, F)$$
$$u \longmapsto b(-, u)$$

is an isomorphism

Definition Two bilinear forms $b_1 : V_1 \times V_1 \longrightarrow F$ $b_2 : V_2 \times V_2 \longrightarrow F$

are isometric $\simeq$ if $\exists$ $\phi : V_1 \longrightarrow V_2$ F-linear isomorphism

$\ni$ $b_2(\phi(u), \phi(v)) = b_1(u,v)$

$b(v,v)$

If $\mathrm{char}(F) \neq 2$. $q : V \longrightarrow F$ quadratic form

q

b $\frac{1}{2}(q(x+y) - q(x) - q(y))$

$\left\{ \begin{array}{l} \text{Non-degenerated} \\ \text{symmetric} \\ \text{bilinear forms } / F \end{array} \right\}$

$b : V \times V \longrightarrow F$

$(x_1, \dots, x_n) M \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$

$\updownarrow$

$\left\{ \begin{array}{l} \text{Symmetric matrices } M \\ \det M \neq 0 \quad / F \\ \Rightarrow \\ \text{Diagonalizable} \end{array} \right\}$

$\left(b(v_i, v_j) \right)$

We denote by $\mathrm{Isom}(F)$ the set of isometric classes of non-degenerated symmetric bilinear forms.

Definition

Orthogonal sum

M_1, M_2 two symmetric non-deg. matrices.

$$M_1 \oplus M_1 = \begin{pmatrix} M_1 & 0 \\ 0 & M_2 \end{pmatrix}$$

$$b \perp b' : (V \oplus V') \times (V \oplus V') \longrightarrow F$$

$$b \perp b' \big((x, x'), (y, y') \big) = b(x,y) + b'(x', y').$$

$$M_1 \otimes M_2 = \Big(a_{ij} \big(M_2 \big) \Big) \quad n \times m$$

$$= \begin{pmatrix} a_1 b_1 & & & & & \\ & \ddots & & & & \\ & & a_1 b_n & & & \\ & & & \ddots & & \\ & & & & a_n b_1 & \\ & & & & & \ddots \; a_n b_n \end{pmatrix}$$

$$b \otimes b' : (V \otimes V') \times (V \otimes V') \longrightarrow F$$

$$\big((x \otimes x'), (y \otimes y')\big) \longmapsto b(x,y) \cdot b'(x',y')$$

$(\mathrm{Isom}(F), \oplus, \otimes)$ forms a semi-ring

Grothendieck-Witt group. $GW(F)$:

$$\begin{tikzcd}
(\mathrm{Isom}(F), \oplus) \arrow[rr] \arrow[d, "\exists\, i"'] & & M \\
GW(F) \arrow[urr, dashed, "\exists !"'] & &
\end{tikzcd}$$

$$GW(F) = (\mathrm{Isom}(F), \oplus)^{gp}$$

$(b_1, b_2) \sim (b_1', b_2') \Leftrightarrow \exists d \quad b_1 \perp b_2' \perp d = b_1' + b_2$

For $a \in F^\times$, we denote by $\langle a \rangle \in GW(F)$

$$b_a : \begin{cases} F \times F \longrightarrow F \\ (x,y) \longmapsto axy \end{cases} \quad \longleftrightarrow \quad q_{b_a}(x) = b_a(x,x) = ax^2$$

Thm: (4.3 Lam 05)
The abelian group is given by the generators $\langle a \rangle$ $a \in F^\times$ under the following relations

i) $\langle ab^2 \rangle = \langle a \rangle \quad \forall\, a, b \in F^\times$

ii) $\langle a\rangle \langle b\rangle = \langle ab\rangle \quad a, b \in F^\times$

iii) $\langle a\rangle + \langle -a\rangle = \underbrace{\langle 1\rangle + \langle -1\rangle = h}_{\text{hyperbolic plane}}$

iv) $\langle a\rangle + \langle b\rangle = \langle a+b\rangle + \langle (a+b)ab\rangle$

$\forall\ a, b \in F^\times \qquad a+b \in F^\times$

The Witt-ring

$$W(F) := GW(F) / \langle h\rangle \qquad h = \langle 1\rangle + \langle -1\rangle = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$GW(F) \xrightarrow{\text{rank}} \mathbb{Z}$$

$$(b_1, b_2) \longmapsto \operatorname{rank} b_1 - \operatorname{rank} b_2$$

$$W(F) \xrightarrow{rk} \mathbb{Z}/2\mathbb{Z}$$

$q = q_h \perp q_a$

anisotropic

$q_h = m \cdot H$

$$\begin{array}{ccccc} & & GW(F) & \xrightarrow{rk} & \mathbb{Z} \\ & & \downarrow & & \downarrow \\ I(F) & \to & W(F) & \xrightarrow{rk} & \mathbb{Z}/2 \end{array}$$

Fundamental Ideal

§2. Milnor-Witt K-theory

Definition: F field.

The Milnor-Witt K-theory of F is the $\mathbb{Z}$-graded associative ring with unit

$$K_*^{MW}(F)$$

generated by the symbols.

- $[u]$ $u \in F^\times$ of degree $+1$
- η of degree -1

subject to the relations.

1) (Steinberg relation)
For $a \in F^\times \setminus \{1\}$:
$$[a]\cdot[1-a]=0$$

2) $(a,b) \in (F^\times)^2$:
$$[ab] = [a] + [b] + \eta[a]\cdot[b]$$
$\eta = 0$
$$\{a.b\} = \{a\} + \{b\}$$
$\Rightarrow$ additive

3) $u \in F^\times$: $\quad [u]\cdot\eta = \eta\cdot[u]$

4) Set $h := \eta\cdot[-1] + 2$

$$\eta\cdot h = 0$$
$$0 = \eta\cdot(\eta\cdot[-1]+2)$$

Relation of $K_*^{MW}(F)$ with other mathematical objects.

a) $K_*^{MW}(F)/(\eta) = K_*^{M}(F)$

$[a_1, \dots, a_n] \longmapsto \{a_1, \dots, a_n\}$

$\shortparallel \qquad\qquad\qquad \shortparallel$

$[a_1]\cdot[a_2]\cdots[a_n] \qquad \{a_1\}\cdot\ldots\cdot\{a_n\}.$

b) $GW(F) \longrightarrow K_0^{MW}(F)$

$\langle a\rangle \longmapsto 1+\eta[a]$

is an epimorphism.

$\langle a\rangle\cdot\langle b\rangle \longmapsto (1+\eta[a])\cdot(1+\eta[b])$

$= 1+\eta[ab]$

From 2), 3) in MW-relations.

Note: $1+\langle -1\rangle = 1+(1+\eta[-1]) = h$

in fact it is a isomorphism

Original definition of K_*^{MW}

Let $I(F)$ be the fundamental ideal in $W(F)$

$$I^n(F) := (I(F))^n \quad \forall \ n > 0$$

$$I^n(F) := W(F) \quad \forall \ n \leq 0$$

$\forall \ n \geq 0$

$$\begin{tikzcd}
J^n(F) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & K_n^M(F) \arrow[d] \\
I^n(F) \arrow[r] & I^n(F)/I^{n+1}(F)
\end{tikzcd}$$

$$J^n(F) := W(F) \qquad n < 0$$

$$K_*^{MW}(F) \cong J^*(F)$$

In particular

$$\boxed{K_0^{MW}(F) \cong GW(F)}$$

The vertical right map is given by

$$\{a_1, \dots, a_n\} \longrightarrow \langle\langle a_1, \dots, a_n \rangle\rangle \mod I^{n+1}(F)$$

$$\langle\langle a_1, \dots, a_n \rangle\rangle := \langle -1, a_1 \rangle \otimes \dots \otimes \langle -1, a_n \rangle$$

Pfister form

Lemma: $\epsilon := -\langle -1 \rangle \in K_0^{MW}(F)$

$\forall\, n \in \mathbb{Z}$ let

$$n_\epsilon := \begin{cases} \sum_{i=1}^{n} \langle (-1)^{i-1} \rangle & \text{if } n > 0 \\ 0 & \text{if } n = 0 \\ \epsilon \sum_{i=1}^{-n} \langle (-1)^{i-1} \rangle & \text{if } n < 0 \end{cases}$$

The the following properties are satisfied

1.- $a \in F^\times$ $[a][-a] = 0$

and $\langle a \rangle + \langle -a \rangle = h$

2.- $\forall$ $a \in F^\times$ we have

$$[a]\cdot[a] = [a]\cdot[-1]$$
$$= \epsilon[a][-1]$$
$$= [-1]\cdot[a]$$
$$= \epsilon[-1][a]$$

3.- $a, b \in F^\times$ $[a]\cdot[b] = \epsilon[b][a]$

4.- $\langle a^2 \rangle = 1.$

<u>**Corollary:**</u> The $K_0^{MW}(F)$ - algebra $K_*^{MW}(F)$ is ϵ - graded commutative

$\alpha \in K_n^{MW}(F)$ $\beta \in K_m^{MW}(F)$

$$\alpha \cdot \beta = (\epsilon)^{n\cdot m} \beta \cdot \alpha$$

§3. Residue homomorphisms

Fundamental tool to define Rost - Schmid complexes.

F field.

$v : F \longrightarrow \mathbb{Z} \cup \{\infty\}$ a discrete valuation

$\mathcal{O}_v$ valuation ring

$\mathfrak{m}_v$ maximal ideal
$\parallel$
(π_v) π_v uniformizer

k_v residue field

3.1 Thm (Non-canonical residue morphism)

March 12, Thm 3.15

$\exists !$ homomorphism of graded abelian groups

$$\partial_v^{\pi} : K_*^{MW}(F) \longrightarrow K_{*-1}^{MW}(k(v))$$

of degree -1

commutes with the mult. by η.

and satisfies.

1) $\partial_v^{\pi}([\pi, u_1, \ldots, u_n]) = [\bar{u}_1, \ldots, \bar{u}_n]$

2) $\partial_v^{\pi}([u_1, \ldots, u_n]) = 0$

$\forall \quad u_1, \ldots, u_n \in \mathcal{O}_v^{\times}$

Where $[\pi, u_1, \ldots, u_n] = [\pi]\cdot[u_1]\cdots[u_n]$.

$\bar{u}_i \in \mathcal{O}_v/m_v = k(v)$

For $n=0$ $\quad \partial_v^{\pi}([\pi]) = 1$.

$\partial_v^{\pi}(1) = 0$

Remarks:

1.- If ∂_v^{π} exists $\Rightarrow$ it is. unique

$[a_1, \ldots, a_n] \in K_n^{MW}(F)$

$a_i = u_i \pi^{n_i} \qquad u_i \in \mathcal{O}_v^{\times} \qquad n_i \in \mathbb{Z}$

$[\pi^{n_i}] = (n_i)_\varepsilon [\pi]$

$[\pi, \pi] = [\pi, -1]$

$\Rightarrow$ By the graded commutativity of MW-K-theory:

$[a_1, \dots, a_n]$ is sum of symbols of the form

- $\eta^m [\pi, u_1, \dots, u_{n+m-1}]$
- $\eta^m [u_1, \dots, u_{n+m}]$

for some $m \in \mathbb{N}$

the image under ∂_v^π is determined by 1) and 2)

2.- ∂_v^π depends not only on v, but also on the choice of the uniformizer π.

Let $\pi' = u\pi$ another uniformizer $u \in \mathcal{O}_v^\times$

By construction:

$\partial_v^\pi([\pi, -1]) = [-1]$

$$\partial_v^{\pi'}([\pi, -1]) = \partial_v^{\pi'}([u^{-1}\pi', -1])$$

$$= \partial_v^{\pi'}\left([u^{-1}, -1] + [\pi', -1] + \eta[u^{-1}, \pi', -1]\right)$$

$$= \underbrace{\overset{0}{\cancel{\partial_v^{\pi'}([u^{-1}, -1]}}}_{2.} + \underbrace{[\pi', -1] + \eta[\pi', u^{-1}, -1])}_{1.}$$

$$= [-1] + \eta[u^{-1}, -1]$$

$$= \langle u^{-1}\rangle [-1]$$

In general $\quad [-1] \neq \langle u^{-1}\rangle [-1]$

For instance for $\quad k(v) = \mathbb{R}$

$$K_1^{MW}(\mathbb{R}) \longrightarrow I(\mathbb{R})$$

$$\begin{array}{ccc} [-1] & \neq & \langle -1\rangle [-1] \\ \| & & \| \\ \langle -1\rangle & & \langle -1\rangle\langle -1\rangle = \langle 1\rangle \end{array}$$

Existence Follows from the following lemma:

Serre's method ξ variable of degree -1

take $K^{MW}(\kappa(v))[\xi]$ such that. $\xi^2 = \xi[-1]$

Lemma: The map.

$$\mathbb{Z} \times \mathcal{O}_v^\times = F^\times \longrightarrow K_*^{MW}(\kappa(v))[\xi]$$

$$\pi^n . u \longmapsto \theta_\pi(\pi^n u) := [\bar{u}] + \langle n_\varepsilon \langle \bar{u} \rangle\rangle \cdot \xi$$

and $\eta \longmapsto \eta$

satisfies the relations of MW K-theory

an induce a morphism of graded rings

$$\theta_\pi : K_*^{MW}(F) \longrightarrow K_*^{MW}(\kappa(v))[\xi]$$

an induce a morphism of graded rings

$$\Theta_\pi : K_*^{MW}(F) \longrightarrow K_*^{MW}(\kappa(v))[\xi]$$

For $a \in K_n^{MW}(F)$, we set:

$$\Theta_\pi(a) := s_v^\pi(a) + \partial_v^\pi(a) \cdot \xi$$

Boundary map. $\partial_v^\pi : K_*^{MW}(F) \longrightarrow K_{*-1}^{MW}(\kappa(v))$

Specialization $s_v^\pi : K_*^{MW}(F) \longrightarrow K_*^{MW}(\kappa(v))$

Remark: $\partial_v^\pi([\pi]) = 1_\varepsilon \cdot \langle 1 \rangle$
$= \langle 1 \rangle = 1 \in K_0^{MW}(\kappa(v))$

$\partial_v^\pi[u] = 0$

$\partial_v^\pi[\pi, u] = \partial_v^\pi \cdot ([\pi][u])$
$= \partial_v^\pi([\pi]) \cdot [\bar{u}] + [\pi] \cdot \cancel{\partial_v^\pi [u]}$
$= [\bar{u}]$

3.2 Canonical residue morphisms.

Since ∂_v^{π} depends on π

we need to introduce

<u>Def</u>: (twisted MW-K-theory)

- $\underbrace{\mathbb{Z}[F^\times]}_{\text{group-ring}}$ is the free abelian group $\bigoplus_{f\in F^\times} \mathbb{Z}\lambda_f$ associated to $F^\times$ with the following product

$$\Big(\sum_{f\in F^\times} n_f \lambda_f\Big)\Big(\sum_{g\in F^\times} m_g\lambda_g\Big) = \sum_{h\in F^\times}\Big(\sum_{\substack{f,g\in F^\times\\ fg=h}} n_f m_g\Big)\lambda_h$$

- L a F-vector space dim 1

$$\mathbb{Z}[L\setminus\{0\}] = \bigoplus_{e\in L\setminus\{0\}} \mathbb{Z}\,\xi_e$$

with the scalar product

$$\Big(\sum_{f \in F^\times} n_f \lambda_f\Big) \cdot \Big(\sum_{g \in L \setminus \{0\}} m_g \xi_g\Big)$$

$$= \sum_{h \in L \setminus \{0\}} \Big(\sum_{\substack{f \in F^\times \\ g \in L \setminus \{0\} \\ f \cdot g = h}} n_f \, m_g \Big) \xi_h$$

- $m \in \mathbb{Z}$ $\quad L \quad$ F - v.space

$\dim(L) = 1$

L - twisted m-th Milnor - Witt K-theory abelian group of F

$$K^{MW}_m(F, L) := K^{MW}_m(F) \otimes_{\mathbb{Z}[F^\times]} \mathbb{Z}[L \setminus \{0\}]$$

Key observation: if we fix an isomorphism $L \overset{\varphi}{\cong} F$.

$$\Rightarrow \quad K_m^{MW}(F, L) \cong K_m^{MW}(F)$$

But it is not canonical !!

unless $L = F$

<u>**Definition**</u> (The canonical residue morphism)

$$\partial_v : K_*^{MW}(F) \longrightarrow K_{*-1}^{MW}\left(\kappa(v), (m_v/m_v^2)^\vee\right)$$

is given by:

$$\partial_v = \partial_v^{\pi} \otimes (\bar{\pi})^*$$

$(m_v/m_v^2)^\vee$ is the dual of the $\kappa(v)$ - vec. space m_v/m_v^2

$\bar{\pi} = [\pi] \in m_v/m_v^2$.

$(\bar{\pi})^*$ the dual basis of $(\bar{\pi})$

<u>Rmk:</u> ∂_v does not depend on the choice of π

If π' is another uniformizer

$$\pi' = u\pi$$

$$\partial_v^{\pi} \otimes \overline{\pi}^* = \langle \bar{u} \rangle \partial_v^{\pi'} \otimes \overline{\pi}^*$$

$$= \partial_v^{\pi'} \otimes \overline{u\pi}^*$$

$$= \partial_v^{\pi'} \otimes \overline{\pi'}^*$$

3.3 The twisted canonical residue morphism

Definition:

Let L be a $\mathcal{O}_v$-mod.

$\operatorname{rank}(L) = 1$

$$\partial_{v,L} : K_*^{MW}(F, L \otimes_{\mathcal{O}_v} F) \to K_{*-1}^{MW}\left(K(v), \left(m_v / m_v^2\right)^{\vee} \otimes_{K(v)} (L \otimes_{\mathcal{O}_v} K(v))\right)$$

is the unique morphism of graded groups such that for all

$\alpha \in K_*^{MW}(F)$

$l \in L$

$$\partial_{v,L}(\alpha \otimes (l \otimes 1)) = \partial_v^{\pi}(\alpha) \otimes (\bar{\pi}^* \otimes (l \otimes 1))$$

Explicit formula for ∂_v^{π} :

Notation: Odd characteristic

$$\chi^{odd} : \begin{cases} \mathbb{Z} \longrightarrow \{0,1\} \\ m \longmapsto \chi^{odd}(m) = \begin{cases} 1 & m \text{ odd} \\ 0 & m \text{ even} \end{cases} \end{cases}$$

(Lemarie Phd thesis)

Theorem 2.46. For all $n \leq 0$, $m \in \mathbb{Z}$ and $u \in \mathcal{O}_v^*$:

$$\partial_v^{\pi}(\langle \pi^m u \rangle \eta^{-n}) = \langle \overline{u} \rangle \eta^{-n+1} \chi^{\text{odd}}(m)$$

For all $n \geq 1$, $m_1, \ldots, m_n \in \mathbb{Z}$ and $u_1, \ldots, u_n \in \mathcal{O}_v^*$:

$$\partial_v^{\pi}([\pi^{m_1} u_1, \ldots, \pi^{m_n} u_n]) =$$

$$\sum_{l=0}^{n-1} \sum_{\substack{J \subset \{1,\ldots,n\}, |J|=l \\ J=\{j_1 < \cdots < j_l\}}} ((-1)^{\sum_{i=1}^{l} n-l+i-j_i} \prod_{k \in \{1,\ldots,n\} \setminus J} m_k)_{\epsilon} [\underbrace{-1, \ldots, -1}_{n-1-l \text{ terms}}, \overline{u_{j_1}}, \ldots, \overline{u_{j_l}}]$$

$$+ \sum_{p=1}^{n} \sum_{l=p}^{n} \sum_{\substack{J \subset \{1,\ldots,n\}, |J|=l \\ J=\{j_1 < \cdots < j_l\}}} (\sum_{\substack{I \subset \{1,\ldots,l\} \\ |I|=p}} \eta^p \chi^{\text{odd}}(\prod_{i \in I} m_{j_i} \times \prod_{k \in \{1,\ldots,n\} \setminus J} m_k))[\underbrace{-1, \ldots, -1}_{n-1+p-l \text{ terms}}, \overline{u_{j_1}}, \ldots, \overline{u_{j_l}}]$$

For $n = 1$

$$\partial_v^{\pi}([\pi^m u]) = m_{\epsilon} + \eta \, \chi^{odd}(m) [\bar{u}]$$

$$= \langle \bar{u} \rangle m_{\epsilon}$$

$n < 0$

$$\partial_v^{\pi}(\langle \pi^m u \rangle \eta^{-n}) = \langle \bar{u} \rangle \eta^{-n+1} m_{\epsilon}$$

§4. Rost - Schmid complexes

F perfect field.

X : smooth finite type schema / $\mathrm{Spec}(F)$

<u>Def</u> (Determinant of a locally free module)

V locally free $\mathcal{O}_X$-mod.

rank $(V) = r$

$\det(V) = \Lambda^r(V)$ r-th exterior product

<u>Notation:</u> • $X^{(i)}$ points of codim $(x, X) = i$

$X^{(i)} = \emptyset$ if $i < 0$ or $\dim(X) < i$

- $\mathcal{N}_{x/X}$ normal sheaf of x in X i.e.

$$\mathcal{N}_{x/X} = \left(\mathfrak{m}_{X,x} / \mathfrak{m}_{X,x}^2 \right)^\vee$$

$$\mathfrak{m}_{X,x} \subset \mathcal{O}_{X,x}$$

$$\nu_x := \det(\mathcal{N}_{x/X})$$

- $\mathcal{L}|_{\{x\}} := \mathcal{L}_{\{x\}} \otimes_{\mathcal{O}_{X,\{x\}}} \kappa(x)$

<u>Definition</u> For $j \in \mathbb{Z}$, $\mathcal{L}$ invertible $\mathcal{O}_X$-mod.

The Rost-Schmid complex associated to $X, j, \mathcal{L}$

$$C(X, \underline{K}_j^{MW}\{\mathcal{L}\})$$

is given by.

$$\cdots \longrightarrow C^i(X, \underline{K}_j^{MW}\{L\}) \xrightarrow{\partial^i_{X,j,L}} C^{i+1}(X, \underline{K}_j^{MW}\{L\})$$
$$\downarrow$$
$$\vdots$$

where

$$C^i(X, \underline{K}_j^{MW}\{L\}) :=$$

$$\bigoplus_{x \in X^{(i)}} K^{MW}_{j-i}\left(k(x), \nu_x \otimes_{k(x)} L|_{\{x\}}\right)$$

$\partial^i_{X,j,L}$ is the only morphism of groups.

such that $\forall\ x \in X^{(i)}$

$$k_x \in K^{MW}_{j-i}\left(k(x), \nu_x \otimes_{k(x)} L|_{\{x\}}\right)$$

$$k_x \xmapsto{\partial^i_{X,j,L}} \sum_{y \in \{x\}^{(1)}} \partial^x_y(k_x)$$

with.

$$d^x_y : K^{MW}_{j-1}(k(x), \nu_x \otimes_{k(x)} \mathbb{Z}1_{\{x\}}) \longrightarrow K^{MW}_{j-i-1}(k(y), \nu_y \otimes_{k(y)} \mathbb{Z}1_{\{y\}})$$

We denote

$$C(X, \underline{K}^{MW}_j) := C(X, \underline{K}^{MW}_j\{\mathcal{O}_X\})$$

<u>Thm</u> (5.31, [Mor 12])

$$d^{i+1}_{X,j,\mathcal{L}} \circ d^i_{X,j,\mathcal{L}} = 0$$

§5. The 5 basic maps

Following FELD

5.1 Pushforward

Let $f: X \longrightarrow Y$ F-morphism of schemes

$\mathcal{L}$ invertible $\mathcal{O}_Y$-mod.

$f^*\mathcal{L}$ invertible $\mathcal{O}_X$-mod.

Define

$$f_*: C^i(X, \underline{K}_j\{f^*\mathcal{L}\}) = \bigoplus_{x \in X^{(i)}} K^{MW}_{j-i}(\kappa(x), \nu(x) \otimes_{\kappa(x)} f^*\mathcal{L}|_x)$$

$$\downarrow$$

$$C^i(Y, K_j\{\mathcal{L}\})$$

If $y = f(x)$ $\kappa(x)/\kappa(y)$ finite

$\Rightarrow (d_x)^x_y = \mathrm{cores}_{\kappa(x)/\kappa(y)}$

corestriccion.

$(f_x)^x_y = 0$ otherwise

5.2 (Pull-back)

$f : X \longrightarrow Y$ essentially smooth

$\mathcal{L}$ invertible $\mathcal{O}_Y$-module

$f^*\mathcal{L}$ $\mathcal{O}_X$-mod.

$$f^* : C^i(Y, K_j\{\mathcal{L}\}) \longrightarrow C^i(X, K_j\{f^*\mathcal{L}\})$$

$$\parallel$$

$$\bigoplus_{y \in Y^{(i)}} K^{MW}_{j-i}(\kappa(y), \nu(y) \otimes_{\kappa(y)} \mathcal{L}|_{\{y\}})$$

If $f(x) = y$

$$(f^*)^y_x = \textcircled{H} \circ \operatorname{res}_{\kappa(x)/\kappa(y)}$$

$$\textcircled{H} : \mathcal{J}_{\operatorname{Spec} \kappa(x) / \operatorname{Spec} \kappa(y)} \simeq \mathcal{J}_{X/Y} \times_X \operatorname{Spec} \kappa(x)$$

$(f^*)_x^y = 0$ otherwise

If X non-connected.

take the sum over each component.

5.3 Multiplication by units.

$a_1, \dots, a_n \in \mathcal{O}_X^\times$ global units.

$\mathcal{L}$ $\mathcal{O}_X$-mod invertible

$$[a_1, \dots, a_n] : C^i(X, K\{\mathcal{L}\}) \longrightarrow C^i(X, K\{n\mathbb{A}_X^1 + \mathcal{L}\})$$

$x \in X^{(i)}$

$\rho \in K^{MW}_{j-i}(\kappa(x), \nu(x) \otimes_{\kappa(x)} \mathcal{L}_{\kappa(x)})$

if $x = y$

$$[a_1, \dots, a_n]^x_y(e) = \oplus \left((-1)^{np} [a_1(x), \dots, a_n(x)] \cdot e \right)$$

$$\eta \cdot A^1_{K(x)} + \Omega_{K(x)/k} \overset{\oplus}{\cong} \Omega_{K(x)/k} + \eta \cdot A^1_{K(x)}$$

$$[a_1, \dots, a_n]^x_y(e) = 0 \quad \text{otherwise}$$

5.4 Multiplication by η.

$$\eta : C^i(X, K^{MW}\{1\}) \longrightarrow C^i(X, K^{MW}(-A^1_x + 1))$$

if $x = y$. $\quad \eta^x_y(e) = \gamma_\eta(e)$

$$\eta^x_y(e) = 0 \quad \text{otherwise}$$

Where γ_η is given in the

data below

Definition 2.3.1.1. A Milnor-Witt cycle premodule M (also written: MW-cycle premodule) is a functor from $\mathfrak{F}_k$ to the category **Ab** of abelian groups with the following data (D1),..., (D4) and the following rules (R1a),..., (R4a).

D1 Let $\varphi : (E, \mathcal{V}_E) \to (F, \mathcal{V}_F)$ be a morphism in $\mathfrak{F}_k$. The functor M gives a morphism $\varphi_* : M(E, \mathcal{V}_E) \to M(F, \mathcal{V}_F)$.

D2 Let $\varphi : (E, \mathcal{V}_E) \to (F, \mathcal{V}_F)$ be a morphism in $\mathfrak{F}_k$ where the morphism $E \to F$ is *finite*. There is a morphism $\varphi^* : M(F, \Omega_{F/k} + \mathcal{V}_F) \to M(E, \Omega_{E/k} + \mathcal{V}_E)$.

D3 Let $(E, \mathcal{V}_E)$ and $(E, \mathcal{W}_E)$ be two objects of $\mathfrak{F}_k$. For any element x of $\underline{\mathrm{K}}^{MW}(E, \mathcal{W}_E)$, there is a morphism

$$\gamma_x : M(E, \mathcal{V}_E) \to M(E, \mathcal{W}_E + \mathcal{V}_E)$$

so that the functor $M(E, -) : \mathfrak{V}(E) \to \mathbf{Ab}$ is a left module over the lax monoidal functor $\underline{\mathrm{K}}^{MW}(E, -) : \mathfrak{V}(E) \to \mathbf{Ab}$ (see [Yet03, Definition 39]; see also remarks below).

D4 Let E be a field over k, let v be a valuation on E and let $\mathcal{V}$ be a virtual projective $\mathcal{O}_v$-module of finite type. Denote by $\mathcal{V}_E = \mathcal{V} \otimes_{\mathcal{O}_v} E$ and $\mathcal{V}_{\kappa(v)} = \mathcal{V} \otimes_{\mathcal{O}_v} \kappa(v)$. There is a morphism

$$\partial_v : M(E, \mathcal{V}_E) \to M(\kappa(v), -\mathcal{N}_v + \mathcal{V}_{\kappa(v)}).$$

R1a Let φ and ψ be two composable morphisms in $\mathfrak{F}_k$. One has

$$(\psi \circ \varphi)_* = \psi_* \circ \varphi_*.$$

R1b Let φ and ψ be two composable finite morphisms in $\mathfrak{F}_k$. One has

$$(\psi \circ \varphi)^* = \varphi^* \circ \psi^*.$$

R1c Consider $\varphi : (E, \mathcal{V}_E) \to (F, \mathcal{V}_F)$ and $\psi : (E, \mathcal{V}_E) \to (L, \mathcal{V}_L)$ with φ finite and ψ separable. Let R be the ring $F \otimes_E L$. For each $p \in \operatorname{Spec} R$, let $\varphi_p : (L, \mathcal{V}_L) \to (R/p, \mathcal{V}_{R/p})$ and $\psi_p : (F, \mathcal{V}_F) \to (R/p, \mathcal{V}_{R/p})$ be the morphisms induced by φ and ψ. One has

$$\psi_* \circ \varphi^* = \sum_{p \in \operatorname{Spec} R} (\varphi_p)^* \circ (\psi_p)_*.$$

R2 Let $\varphi : (E, \mathcal{V}_E) \to (F, \mathcal{V}_F)$ be a morphism in $\mathfrak{F}_k$, let x be in $\underline{\mathrm{K}}^{MW}(E, \mathcal{W}_E)$ and y be in $\underline{\mathrm{K}}^{MW}(F, \Omega_{F/k} + \mathcal{W}'_F)$ where $(E, \mathcal{W}_E)$ and $(F, \mathcal{W}'_F)$ are two objects of $\mathfrak{F}_k$.

R2a We have $\varphi_* \circ \gamma_x = \gamma_{\varphi_*(x)} \circ \varphi_*$.

R3e Let E be a field over k, v be a valuation on E and u be a unit of v. Then

$$\partial_v \circ \gamma_{[u]} = \gamma_{\varepsilon[\overline{u}]} \circ \partial_v \text{ and}$$
$$\partial_v \circ \gamma_\eta = \gamma_\eta \circ \partial_v.$$

R4a Let $(E, \mathcal{V}_E) \in \mathfrak{F}_k$ and let Θ be an endomorphism of $(E, \mathcal{V}_E)$ (that is, an automorphism of $\mathcal{V}_E$). Denote by Δ the canonical map[5] from the group of automorphisms of $\mathcal{V}_E$ to the group $\mathbf{K}^{MW}(E, 0)$. Then

$$\Theta_* = \gamma_{\Delta(\Theta)} : M(E, \mathcal{V}_E) \to M(E, \mathcal{V}_E).$$

5.5 Boundary map.

X / k scheme of finite type

$$Z \xrightarrow{i} X \xleftarrow{j} U$$

$$\operatorname{Im}(j) = X \setminus \operatorname{im}(i)$$

Z, X, U smooth finite type of pure dimension

d_Z, d_X resp.

$$\mathcal{V}_Z = \det(N_{Z/X})$$
$$N_{Z/X} = (J_Z / J_Z^2)^\vee$$

the boundary map is given by

$$\partial : C^{n+d_x-d_z}(U, \underline{K}^{MW}_{m+d_x-d_z}) \longrightarrow C^{n+1}(Z, \underline{K}^{MW}_m\{\nu_Z\})$$

$$\partial := i^* \circ d^{m+d_x-d_z}_{X, m+d_z-d_x} \circ j^*$$