Quadratic intersection theory and motivic linking

Motives summer seminar 2025

Lecture 1 is an overview, **Lectures 2-6** present Rost's cycle complexes and focus on the Rost complex associated to Milnor K-theory (and on "classical" intersection theory), **Lecture 7** presents motivic spheres and motivation for Milnor-Witt K-theory, **Lectures 8-11** present the Rost-Schmid complex associated to Milnor-Witt K-theory (and quadratic intersection theory), **Lecture 12** is a background lecture on (classical) knot theory and projective knot theory which will be useful for the last two lectures and **Lectures 13-14** present motivic knot theory, specifically motivic linking.

I Lecture 1 (April 7) An overview

An overview of this seminar.

II Lecture 2 (April 14) Milnor K-theory and the Rost complex

The speaker will present Milnor K-theory K^{M} and Rost complexes, with a particular focus on the Rost complex associated to K^{M}. This is the first lecture in a series of lectures about the Rost complex; our goal is to give intuition in this simpler setting to pave the way for the lectures on the Rost-Schmid complex (which is the main tool for motivic linking).

Main references: Sections 1-2 of Milnor's article [Mil70] and Sections 1-3 of Rost's article [Ros96].

III Lecture 3 (April 23) Rost groups and Chow groups

After recalling the classical definition of Chow groups, the speaker will present Rost groups. The Rost group $A^i(X; M, j)$ (also denoted $A^i(X, j)$ when $M = K^{\mathrm{M}}$), where i and j are integers, is the i-th cohomology group of the Rost complex $C^*(X; M, j)$. This generalises Chow groups as the Chow group $\mathrm{CH}^i(X)$ is $A^i(X; K^{\mathrm{M}}, i)$ (or $A^i(X, i)$ for short). The speaker will end the talk by explicitly computing some Rost groups.

Main references: Sections 4-5 of Rost's article [Ros96].

Additional references: the books [Ful98] by Fulton and [EH16] by Eisenbud and Harris.

IV Lecture 4 (April 30) An explicit proof of the homotopy property

The speaker will present the construction of a homotopy inverse of the morphism of Rost complexes induced by an affine bundle. This construction (together with the homotopy axiom) yields the homotopy property for affine bundles (see [Ros96, Proposition 8.6]). More importantly, it yields this property in an explicit manner: this construction takes place in the Rost complex (before the quotienting that gives rise to the Rost groups), which is essential for computations.

Main reference: Section 9 of Rost's article [Ros96].

V Lecture 5 (May 7) Deformation to the normal cone and pull-backs

The speaker will present an important technique, deformation to the normal cone (also known as specialisation to the normal cone), which will then be used to define general pull-backs. These general pull-backs will themselves be used to define intersection products in the following talk.

Main references: Sections 10-12 of Rost's article [Ros96].

Additional references: the books [Ful98] by Fulton and [EH16] by Eisenbud and Harris.

VI Lecture 6 (May 14) Intersection products

After recalling the classical definitions of the intersection product and of the Chow ring, the speaker will present the Rost ring, i.e. the direct sum (over all integers i and j) of the Rost groups $A^i(X; M, j)$ together with Rost's intersection product. The Chow ring is also the direct sum over all integers i of the Chow groups $A^i(X; K^{\mathrm{M}}, i)$ together with Rost's intersection product (which coincides with the classical intersection product), so that it is a subring of the Rost ring associated to Milnor K-theory. If possible, the speaker will finish the talk by giving a formula to compute Rost's intersection product in nice cases and by applying this formula to explicit computations.

Main references: Sections 13-14 of Rost's article [Ros96].

Additional references: the books [Ful98] by Fulton and [EH16] by Eisenbud and Harris.

VII Lecture 7 (May 21) From spheres to motivic spheres

After giving some background on the stable homotopy groups of (classical) spheres, the speaker will present motivic spheres and their stable homotopy groups. This will give an idea of the importance of Milnor-Witt K-theory (thanks to Morel's theorem), which will be introduced in the following talk. The speaker will end this talk by presenting smooth models of motivic spheres, which will be used in the final two lectures.

Main reference: the article [ADF16] by Asok, Doran and Fasel.

Additional references: Lecture 10 "The Suspension Theorem and Homotopy Groups of Spheres" in the book [FF16] by Fomenko and Fuchs, the article [IWX23] by Isaksen, Wang and Xu and Morel's book [Mor12].

VIII Lecture 8 (May 28) Milnor-Witt K-theory and the Rost-Schmid complex

After giving some background on the Witt ring and the Grothendieck-Witt ring, the speaker will present Milnor-Witt K-theory K^{MW} and the Rost-Schmid complex (associated to K^{MW}). This is the first lecture in a series of lectures about the Rost-Schmid complex (paving the way to the study of motivic linking). The speaker will end this talk by presenting the counterparts to the "four basic maps" discussed in Lecture 2, as well as the fifth basic map: multiplication with η.

Main references: Chapters 2-3 of Lemarié--Rieusset's PhD thesis [Lem23], Morel's book [Mor12] (especially Chapters 3-5) and Fasel's lecture notes [Fas20].

Additional references: Barge and Morel's article [BM00], Morel's lecture notes [Mor03], Schmid's PhD thesis [Sch98] and Feld's PhD thesis [Fel21].

IX Lecture 9 (June 4) Rost-Schmid groups and Chow-Witt groups

The speaker will present Rost-Schmid groups (and in particular Chow-Witt groups). The Rost-Schmid group $H^i(X, \underline{K}_j^{\mathrm{MW}}\{\mathcal{L}\})$, where i and j are integers and $\mathcal{L}$ is an invertible $\mathcal{O}_X$-module (a.k.a. line bundle on X), is the i-th cohomology group of the Rost-Schmid complex $C^*(X, \underline{K}_j^{\mathrm{MW}}\{\mathcal{L}\})$. The Rost-Schmid group $H^i(X, \underline{K}_i^{\mathrm{MW}}\{\mathcal{L}\})$ is also denoted $\widetilde{\mathrm{CH}}^i(X, \mathcal{L})$ and called "Chow-Witt group". The speaker will end the talk by explicitly computing some Rost-Schmid groups and, if possible, presenting the construction of a homotopy inverse of the morphism of Rost-Schmid complexes induced by an affine bundle.

Main references: Chapters 2-3 of Lemarié--Rieusset's PhD thesis [Lem23], Morel's book [Mor12] (especially Chapters 3-5) and Fasel's lecture notes [Fas20].

Additional references: The article [BM00] by Barge and Morel, Morel's lecture notes [Mor03], Schmid's PhD thesis [Sch98] and Feld's PhD thesis [Fel21].

X Lecture 10 (June 11) Pull-backs

The speaker will define general pull-backs. These general pull-backs will themselves be used to define intersection products in the following talk.

Main references: Chapters 2-3 of Lemarié--Rieusset's PhD thesis [Lem23], Morel's book [Mor12] (especially Chapters 3-5) and Fasel's lecture notes [Fas20].

Additional references: The article [BM00] by Barge and Morel, Morel's lecture notes [Mor03], Schmid's PhD thesis [Sch98] and Feld's PhD thesis [Fel21].

XI Lecture 11 (June 18) Intersection products

The speaker will present the Rost-Schmid ring, i.e. the direct sum (over all integers i and j and line bundles $\mathcal{L}$ on X) of the Rost-Schmid groups $H^i(X, \underline{K}_j^{\mathrm{MW}}\{\mathcal{L}\})$ together with the quadratic intersection product. The subring of the Rost-Schmid ring whose underlying group is the direct sum of the Chow-Witt groups $H^i(X, \underline{K}_i^{\mathrm{MW}}\{\mathcal{L}\})$ (also denoted $\widetilde{\mathrm{CH}}^i(X, \mathcal{L})$) is called the Chow-Witt ring. If possible, the speaker will finish the talk by giving a formula to compute the quadratic intersection product in nice cases and by applying this formula to explicit computations.

Main references: Chapters 2-3 of Lemarié--Rieusset's PhD thesis [Lem23], Morel's book [Mor12] (especially Chapters 3-5) and Fasel's lecture notes [Fas20].

Additional references: The article [BM00] by Barge and Morel, Morel's lecture notes [Mor03], Schmid's PhD thesis [Sch98] and Feld's PhD thesis [Fel21].

XII Lecture 12 (June 23) Knot theory and projective knot theory

After giving some general background on the linking number, the speaker will present linking in (classical) knot theory and in projective knot theory. Specifically, the speaker will define oriented (classical) knots and links before studying the linking number of an oriented link with two components (first in the most classical case, $\mathbb{S}^1 \sqcup \mathbb{S}^1 \hookrightarrow \mathbb{S}^3$, then in the more general case $\mathbb{S}^p \sqcup \mathbb{S}^q \hookrightarrow \mathbb{S}^{p+q+1}$). The speaker will then define oriented projective knots and links before studying the linking number of an oriented projective link with two components (first in the most classical case, $\mathbb{RP}^1 \sqcup \mathbb{RP}^1 \hookrightarrow \mathbb{RP}^3$, then in the more general case $\mathbb{RP}^p \sqcup \mathbb{RP}^q \hookrightarrow \mathbb{RP}^{p+q+1}$).

Main references: Chapter 10 of the book [ST80] by Seifert and Threlfall, Chapter 1 of Lemarié--Rieusset's PhD thesis [Lem23] and Julia Viro's article [Dro91].

Additional references: The articles [Dro94] and [Vir07] by Julia Viro (whose previous name was Julia Drobotukhina) and the chapter [VV21] by Julia Viro and Oleg Viro.

XIII Lecture 13 (July 2) The linking number and the quadratic linking degree

The speaker will present motivic linking, a theory in algebraic geometry which is a counterpart to classical linking. Specifically, the speaker will define quadratic linking degrees which are counterparts to the linking number and describe how two disjoint closed F-subschemes of an F-scheme are "linked" (i.e. intertwined in some sense), with F a perfect field. The quadratic linking degrees are thus called because their definition uses quadratic intersection theory and because, in some interesting settings, they take values in either the Witt group $\mathrm{W}(F)$, the Grothendieck-Witt group $\mathrm{GW}(F)$ or the first Milnor-Witt K-theory group $K_1^{\mathrm{MW}}(F)$ (depending on the setting).

Main references: Chapters 4-5 of Lemarié--Rieusset's PhD thesis [Lem23], her article [Lemb] and her preprint [Lema].

XIV Lecture 14 (July 9) Computations of the quadratic linking degree

The speaker will present two particularly interesting settings for motivic linking: the setting $\mathbb{A}^2_F \setminus \{0\} \sqcup \mathbb{A}^2_F \setminus \{0\} \hookrightarrow \mathbb{A}^4_F \setminus \{0\}$ and the setting $\mathbb{P}^1_F \sqcup \mathbb{P}^1_F \hookrightarrow \mathbb{P}^3_F$, with F a perfect field. The speaker will then compute quadratic linking degrees on examples taking place in these settings, highlighting what information is gained on how these closed F-subschemes of $\mathbb{A}^4_F \setminus \{0\}$ or of $\mathbb{P}^3_F$ are linked. The speaker will end the talk by a discussion of other settings and examples which are of particular interest for motivic linking.

Main references: Chapters 6-7 of Lemarié--Rieusset's PhD thesis [Lem23], her article [Lemb] and her preprint [Lema].

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