§ Ordered Fields

We recall some elementary facts

- An ordered field is a field F with a strict total order such that

$$a<b \Rightarrow a+c<b+c \quad \forall c \in F$$

and $ac<bc \quad \forall c>0 \in F$

This implies

- $\sum_1^r p_i x_i^2 > 0$ for all $p_i, x_i > 0$, $r \geq 1$
- -1 is not a sum of squares

- A real closed field is an ordered field $(F, <)$ such that

- $x>0 \Rightarrow x=y^2$ for some $y \in F$

and

- $P(t) \in F[t]$ a polynomial of odd degree then P has a root in F

equivalently $F[x]/(x^2+1)$ is algebraically closed

Note. An ordered field has characteristic 0

- $\mathbb{Q}, \mathbb{R}$ are ordered fields with the usual order
- these are the unique such

- Given ordered fields $(F,<)$, $(F',<')$ there is at most one ordered field homomorphism $(F,<) \to (F',<')$

- Every ordered field $(F,<)$ admits a real closure $(R,<)$, i.e. $(F,<) \to (R,<)$, $(R,<)$ is real closed and R is algebraic over F

(Artin–Schreier)

- Given

$$\begin{tikzcd} (K,<) \arrow[r] \arrow[d] & (R,<) \arrow[d, dashed, "\exists!"] \\ (L,<) \arrow[r] & (M,<) \end{tikzcd}$$

(amalgamation property)

- ~~two~~ each real closed fields $(R,<)$, $(R',<')$ satisfy the same 1^{st} order statements. (Tarski)

- A subfield K of an ordered field $(R,<)$ acquires the induced order $<|_K$ making $(K,<)$ an ordered field

Real spectrum

Def A a commutative ring. The real spectrum $\mathrm{Sper}(A)$ is the set of pairs $(x, <)$ where $x \in \mathrm{Spec}\,A$ and $<$ is an order on the residue field $k(x)$

makes $(k(x), <)$ an ordered field

Equivalently, pairs $(x,\ k(x) \hookrightarrow (R,<))$
R real closed, where one identifies $k(x) \hookrightarrow (R,<)$ and $k(x) \hookrightarrow (S,<)$

if $\exists$

$$\begin{tikzcd} k(x) \arrow[r, hook] \arrow[d, hook] & (R,<) \arrow[d, hook] \\ (S,<) \arrow[r, hook] & (T,<) \end{tikzcd}$$

Give $\mathrm{Sper}(A)$ the topology: for $a \in A$

have $D(a) = \{(x, <_x) \in \mathrm{Sper}(A[1/a]) \mid \bar{a} >_x 0\}$ $\quad \bar{a} = \mathrm{im}(a)$ in $k(x)$

Then the $\{D(a),\ a \in A\}$ form a basis of open subsets

Note: $A \mapsto \mathrm{Sper}(A)$ extends to a functor

$\mathrm{Sper}: \mathrm{CAlg}^{op} \to \mathrm{Top}$

Def For X a scheme, define

$$\mathcal{R}(X) = \bigcup_{\substack{U = \mathrm{Spec} A \subset X \\ \text{open}}} \mathrm{Sper}(A)$$

eg $\mathcal{R}(\mathrm{Spec} A) = \mathrm{Sper}(A)$

$X \mapsto \mathcal{R}(X)$ extends to a functor $\mathcal{R}(-): \mathrm{Sch} \to \mathrm{Top}$.

Lemma For X finite type over $\mathbb{Z}$, the projection

$$X_{\mathbb{R}} \to X_{\mathbb{Q}} \to X$$

induced homeomorphism $\mathcal{R}(X_{\mathbb{R}}$

Pf Reduce to $X = \mathrm{Spec} A$ $A/\mathbb{Z}$ finite type for $x \in \mathrm{Spec} A$

if $k(x)$ admits an order then $\mathrm{char}\, k(x) = 0$

$\Rightarrow$ $\mathrm{Spec}(A_{\mathbb{Q}}) \hookrightarrow \mathrm{Spec} A$

Given $(x, <)$ in $\mathrm{Sper}(A_{\mathbb{Q}})$ have

$(\mathbb{Q}, <) \to (\mathbb{Q}(x), <)$

$\Big\}$ unique $\qquad \Big\}$ unique $\quad$ gives $\mathrm{Sper}(A_{\mathbb{Q}}) \xrightarrow{\sim} \mathrm{Sper}(A_{\mathbb{R}})$

$(\mathbb{R}, <) \to (\mathbb{R}(x), <)$

<u>Note</u> Sper A is a <u>spectral</u> <u>space</u> i.e. homeomorphic to Spec R for some R. Equivalently a spectral space is a topological space S such that

- S is T_0 (for $x \neq y \in S$ $\exists$ $U^{\text{open}} \ni x$, $y \notin U$ or $\exists$ $V^{\text{open}} \ni y$, $x \notin V$)
- the set of quasi compact open subsets of S is closed under finite intersection and gives basis for the topology
- S is sober: each irreducible closed subset Z has a (unique) generic pt z_{gen} (i.e. $\overline{\{z_{gen}\}} = Z$).

A spectral map of spectral spaces $f: S \to T$ is a continuous map s.t. $f^{-1}(U)$ is quasi compact for each quasi cpt open $U \subset T$.

A subset set Z of a spectral space is constructible if Z is a finite union of subsets of the form $U \cap V^c$, U & V quasi compact opens

Real étale topology

Def X a scheme (separated), Recall the étale site X_{et}

$$\text{obj} = \{ U \to X \mid \text{étale finite type} \}$$

$$\text{cov}_{et}(U) = \{ \{ f_\alpha : U_\alpha \to U \} \mid \coprod_\alpha U_\alpha \to U \text{ is surjective} \}$$
in U_{et}

The real étale site X_{ret} has the same objects as X_{et}

for $U \to X$ in X_{ret}

$$\text{cov}_{ret}(U) = \{ \{ f_\alpha : U_\alpha \to U \} \mid \coprod_\alpha R(U_\alpha) \to R(U) \text{ is surjective} \}$$
in U_{ret}

Schreiber's comparison theorem

For a site $\mathcal{C}$ we have $Shv^{(Ab)}(\mathcal{C})$: category of sheaves of sets (abelian groups)

recall: a morphism of sites ("geometric morphism") $f: \mathcal{D} \to \mathcal{C}$ is given by a functor $f^{-1}: \mathcal{C} \to \mathcal{D}$ such that

- f^{-1} is continuous: f^{-1} maps $Cov_{\mathcal{C}}(X)$ to $Cov_{\mathcal{D}}(f^{-1}X)$ $\forall X$ obj in $\mathcal{C}$
- the corresponding "pullback" functor on sheaves $f_s^{-1}: Shv(\mathcal{C}) \to Shv(\mathcal{D})$ $= f_*$ is exact

Usually write $f^* = a_s$, $f_* = a^s$ & we have the adjunction

$$Hom_{Shv(\mathcal{D})}(f^*\mathcal{F}, \mathcal{G}) = Hom_{Shv(\mathcal{C})}(\mathcal{F}, f_*\mathcal{G})$$

<u>Note</u>: a "geometric morphism" $f: Shv(\mathcal{D}) \to Shv(\mathcal{C})$ (also "morphism of topoi") is given by a pair $f_*: Shv(\mathcal{D}) \to Shv(\mathcal{C})$, $f^*: Shv(\mathcal{C}) \to Shv(\mathcal{D})$

s.t. $f^* \dashv f_*$ & f^* is exact.

so a morphism of sites $f: \mathcal{D} \to \mathcal{C}$ defines a morphism of topoi

$$f: Shv(\mathcal{D}) \to Shv(\mathcal{C})$$

$$\parallel$$

$$(f^* \dashv f_*)$$

To compare $Shv(X_{ret})$ and $Shv(R(X))$

Scheiderer introduces a "comparison" site

X_{com} obj $= \{ (U, W),\ U \to X$ in X_{et}, $W \subset R(U)$ open $\}$

Morp $(U, W) \to (U', W')$ is
$$\begin{matrix} \downarrow & & \downarrow \\ X & & X \end{matrix}$$

$f: U \to U'$ over X s.t.

$R(f)(W) \subset W'$

$Cov((U, W)) = \{ \{ f_\alpha : (U_\alpha, W_\alpha) \to (U, W) \} \mid$

$\{ f_\alpha : U_\alpha \to U \} \in Cov_{et}(U)$

$R(f_\alpha)(W_\alpha) = W \}$

map $X_{ana} \xrightarrow{p_1} X_{et}$ of sites

$$\begin{tikzcd} (U, W) \arrow[d] \arrow[r, mapsto] & U \\ W & \end{tikzcd}$$

$p_2 \swarrow$ $R(U)_{top}$

Theorem (Scheiderer) $p_1 : Shv(X_{ana}) \to Shv(X_{et})$

and $p_2 : Shv(X_{ana}) \to Shv(R(X)_{top})$

are equivalences

Idea for p_1, use the fact that obj of the form

$(U, R(U))$ are cofinal; given

$(V, W \subset R(V)) \Rightarrow U \xrightarrow{f} V$ et

$\wedge$ $w \in W$ s.t. $w \in R(f)(R(U)) \subset W$

idea $U = V(\sqrt{\varepsilon^2 - \sum x_i^2})$ x_i local coords around w

for p2 use the fact that for $U \to V$ étale & open

$R(U) \to R(V)$ is

locally on $R(U)$ a homeomorphism

Proper base change (Deligne / Scholze)

__Theorem__ Let

$$\begin{tikzcd} Y_1 \arrow[r, "g_1"] \arrow[d, "f_1"'] & Y \arrow[d, "f"] \\ X_1 \arrow[r, "g"] & X \end{tikzcd}$$

be a cartesian diagram with f proper

$\forall$ take $\mathcal{F} \in Sh^{Ab}(Y_{ét})$ then the base change map

$$g^*_{ét} R^q f_{ét *} \mathcal{F} \to R^q f_{1,ét*}\, g^*_{1,ét} \mathcal{F}$$

is an isomorphism

Semi-algebraic sets and Delfs's comparison theorem

Take X finite type over $\mathbb{R}$. We have

$$X(\mathbb{R}) \hookrightarrow R(X), \quad \text{continuous}$$

(this all makes sense with $\mathbb{R}$ replaced with a real closed R)

<u>Def</u> let $X = \operatorname{Spec} A$, $A = \mathbb{R}[T_1, \dots, T_n]/I$

A subset $Z \subset X(\mathbb{R})$ is <u>semi algebraic</u> if $\exists$ polynomials P_i, $Q_i \in \mathbb{R}[T_1, \dots, T_n]$ s.t. $i = 1, \dots, r$

$$Z = \{ x \in X(\mathbb{R}) \mid P_i(x) = 0, \ Q_i(x) > 0 \}$$

$X(\mathbb{R}) \subset \mathbb{R}^n$, $(\geq)$

for $X/\mathbb{R}$ finite type, $Z \subset X(\mathbb{R})$ is semi algebraic if $\exists$ affine open cover $X = \bigcup \operatorname{Spec} A_i = U_i$ s.t. $Z \cap U_i(\mathbb{R})$ is semi alg $\forall i$

Give Z the induced topology from $X(\mathbb{R})$

A continuous map $\underset{\underset{X(\mathbb{R})}{\cap}}{Z} \xrightarrow{f} \underset{\underset{Y(\mathbb{R})}{\cap}}{W}$ of semialgebraic sets

is semi algebraic if the graph $\Gamma_f \subset X(\mathbb{R}) \times Y(\mathbb{R})$

is semi algebraic

Note These notions are well-defined, independent of

the choice X and/or Z.

Def $X/_{\mathbb{R}}$ finite type a subset $Z \subset R(X)$

is semi algebraic if we can write $X = \bigcup U_j = \operatorname{Spec} A_j$

and $Z \cap R(U_j)$ is defined by $\qquad A_j = \mathbb{R}[T]/I_j$

$$P_1 = \dots = P_n = 0$$

$$Q_1 > 0, \quad Q_l > 0$$

A

Def $X/\mathbb{R}$ finite type. A sheaf $\mathcal{F}$ on $X(\mathbb{R})$

is constructible if $X(\mathbb{R}) = \coprod Z_i$

$Z_i \subset X(\mathbb{R})$ locally closed, semi-algebraic

such that $\mathcal{F}|_{Z_i}$ is a constant sheaf (of finite sets / of f.gen. ab. gps)

Constructible sheaves on $R(X)$ are defined similarly using the constructible subsets.

A sheaf $\mathcal{F}$ on X_{ret} is constructible if $\exists$

$X = \coprod Z_i$, Z_i locally closed, such that $\mathcal{F}|_{Z_i}$ is locally constant sheaf

(of finite sets / f.gen. ab. gps)

Theorem (Delfs) Let Z be a semi-algebraic subspace of $\mathrm{Spec}(A)$ for $A/\mathbb{R}$ finite type, given the continuous inclusion $Z(\mathbb{R}) \overset{i}{\subseteq} Z$. Let M be a constructible sheaf on Z

Then $i^*: H^*(Z, M) \to H^*_{sing}(Z(\mathbb{R}), i^*M)$

is an isomorphism

<u>Corollary</u> Let X be a finite type $\mathbb{R}$-scheme

$\mathcal{F}$ a constructible sheaf on X_{ret}. Via the

equivalence $Shv(X_{ret}) \underset{\alpha}{\cong} Shv(\mathcal{R}(X))$

we have the corresponding constructible sheaf $\mathcal{R}(\mathcal{F})$ on $\mathcal{R}(X)$

and $i^* \mathcal{R}(\mathcal{F})$ on $X(\mathbb{R})$. Then

$i^* \circ \mathcal{R}(\alpha): H^*(X_{ret}, \mathcal{F}) \to H^*_{sing}(X(\mathbb{R}), i^*\mathcal{R}(\mathcal{F}))$

is an isomorphism

Pf By Scheiderer then $\mathcal{R}(\alpha): H^*(X_{ret}, \mathcal{F}) \to H^*(\mathcal{R}(X), \mathcal{R}(\mathcal{F}))$

is an iso. Then use the decomposition of $\mathcal{R}(\mathcal{F})$,

resp $\mathcal{R}(\mathcal{F})(\mathbb{R})$ into constant sheaves

Deligne's theorem and a spectral sequence to conclude

Bachmann's paper "Recollections on real étale cohomology"

For X a scheme we have the small rét site $X_{rét}$

and also $Sm(X)_{rét}$, uses $Y_{rét}$ for all $Y \to X$ smooth finite type

We have the inclusion $e: X_{rét} \to Sm(X)$

inducing a geometric morphism of sites

$$e^* : Shv(X_{rét}) \rightleftarrows Shv(Sm(X)_{rét}) : r = e_*$$

left Kan extension

i.e. pullback

Lemma 5 e^* is fully faithful and r preserves colimits

Pf on presheaves we have $e^p : PSh(X_{rét}) \rightleftarrows PSh(Sm(X)_{rét}) : e_*$

for $F \in PSh(X_{rét})$ represented by $U \to X$

$e^p(F)(Y \to X) = U \times_X Y \to Y$ and

$e_{\ast} e^{P}(F)\left((U \times_X Y \to Y)(V \to X) \right) = U \times_X V \to V = (U \to X)(V)$

$= F(U)$

but e^{P} & $e_{\ast}$ commute with colimits & $e_{\ast} = e_{P}$

Also $e^{\ast} = a_{\text{ret}} e^{P} \Rightarrow e_{\ast} e^{\ast} = e_{\ast}(a_{\text{ret}} e^{P})$

$= a_{\text{ret}} e_{\ast} e^{P}$ (e_{P} is rad)

$= a_{\text{ret}}(\text{id})$

$= \text{id}$

so $\text{id} \xrightarrow{\sim} e_{\ast} e^{\ast} \Rightarrow e^{\ast}$ is fully faithful

e_{P} preserves colimits in PSh & $e_{P} a_{\text{ret}} = a_{\text{ret}} e_{P}$; a_{ret} is exact

$\Rightarrow e_{\ast}$ preserves colimits

<u>Cor 6</u> Let $f : X \to Y$ be a morphism of schemes

$f^{-1} = f^{-1} : Y_{\text{ret}} \to X_{\text{ret}}$. Then f defines a

morphism of sites $f : X_{\text{ret}} \to Y_{\text{ret}}$ giving $f^{\ast} : \text{Sh}(Y_{\text{ret}}) \rightleftarrows \text{Sh}(X_{\text{ret}}) : f_{\ast}$

a geometric morphism. Moreover $Lf^{\ast} : SH(Y_{\text{ret}}) \to SH(X_{\text{ret}})$

$D(Y_{\text{ret}}) \to D(X_{\text{ret}})$

are t-exact

Pf f^{-1} preserves pullbacks & maps covers to covers $\Rightarrow \underline{f}^{*} \to f_0$

$\Rightarrow$ f^{-1} defines a morphism of sites

$f: X_{et} \to Y_{et}$

The t-exactness of Lf^{*} is Lemma 4 from last lecture

Main theorem

Theorem 8 We have the maps $X \xrightarrow{i_0} X \times \mathbb{A}^1$

and $i_0 \amalg i_1 : X \amalg X \to X \times (\mathbb{A}^1 \setminus \{0\})$. Then for

$F \in Shv(X_{et})$ the maps

$$i_0^{*} : H^{*}_{et}(X \times \mathbb{A}^1, p^{*}F) \to H^{*}_{et}(X, F)$$

$$(i_0, i_1)^{*} : H^{*}_{et}(X \times (\mathbb{A}^1 \setminus 0), p^{*}F) \to H^{*}_{et}(X, F) \oplus H^{*}_{et}(X, F)$$

are iso

Pf By a colimit argument we reduce to the case X finite type $/\mathbb{Z}$, and F constructible

Thm by Scheiderer comparison Thm, replace $(\ )_{ret}$ with $R(-)$ and then replace X with the $X_{\mathbb{Q}}$ $\to X_{\mathbb{R}}$ by lemma 0

Then by Corollary 0, replace $R(-)$ with $(\ )(p)$ where the result is clear □

<u>Theorem 9</u> (Proper base change) Let

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{g} & Y \end{array}$$

be a cartesian diagram with f proper and S, Y Noetherian of finite Krull dimension. Then for $E \in SH(X_{ret})$ (resp $D(X_{ret})$) the canonical base change map

$$g^* Rf_*(E) \to Rf'_* g'^*(E)$$

is a weak equivalence

Pf (for SH) Note that this map is ... because have the Quillen adj $f_\# \dashv f^*$, $f'_\# \dashv f'^*$

<u>Step 0</u> get rid of : This follows in $Rf_\# E$ is in $SH(X_{\text{nét}})$

and g is a map in $X_{\text{nét}}$

<u>Step 1</u> If $X \to Y$ arb. in gen the Quillen adjunct $f_* \dashv f^*$, etc

onto the Postnikov tower for E, gives the spectral

seq'n $E_2^{pq} = R^p f_* \underline{\pi}_{-q} E \Rightarrow \underline{\pi}_{-p-q}(Rf_* E)$

This strongly converges if Y has finite Krull dim & Noetherian $\Rightarrow$ bounded nét dim at all stalks

For this we may take E fibrant

so $Rf_* E = f_* E$ & then we take

$$
\begin{array}{ccccccc}
f_*(\cdots \to & \tau_{\geq n} E & \to & \tau_{\geq n-1} E & \to \cdots & \to E) \\
& \downarrow & & \downarrow & & \\
& \underline{\pi}_n F & & \underline{\pi}_{n-1} F & &
\end{array}
$$

Step 2 For f proper + rel dim $\leq n$, we claim

$R^p f_* F = 0$ for $p > n$, $F \in Shv(Y_{et})$

Pf by Schröer's proper base change thm

+ f
$$\begin{array}{ccc} Y_x & \xrightarrow{i'} & Y \\ \downarrow f_x & & \downarrow f \\ x & \xrightarrow{i} & X \end{array}$$
we have

$$(R^p f_* F)_x = i^*(R^p f_* F)$$

$$= R^p f_{x*} i'^* F$$

$$= 0 \text{ since } \dim Y_{et} + \text{cd } \dim Y_x \leq n$$

(Schröer Thm 7.6)

Step 3 Conclusion The statement on Y' is local so we may assume Y' affine. By colimit argument we may assume Y' is finite type over Y, hence Noeth + finite Krull dim. Then we examine the exchange map between the two strongly convergent spectral sequences (g^* is exact) (g'^* is exact)

$$E_2^{p,q} = g^* R^p f_* \underline{\Pi}_{-q} E \Rightarrow \underline{\Pi}_{-p-q}\, g^* R f_* E$$

$$\downarrow Ex^*_* \qquad\qquad\qquad \downarrow Ex^*_*$$

$$E_2'^{p,q} = R^p f'_* g'^* \underline{\Pi}_{-q} E \Rightarrow \underline{\Pi}_{-p-q}\, R f'_* g'^* E$$

$$g'^* \underline{\Pi}_{-q} E = \underline{\Pi}_{-q}\, g'^* E$$

By Schwede's proper base change theorem the exchange maps on the E_2 terms are all iso's

$\Rightarrow$ theorem. $\square$