Motives WS 24-25
Talk 8

Main theorems

I The results

Thm 35 (+ $\nearrow$ $\downarrow e$): If S is a Noetherian scheme of finite dimension then the functors (which are the canonical ones) in the following (natural in S) commutative diagram are all equivalences.

$$\begin{tikzcd}
 & SH^{S^1}(S)^{\mathrm{r\acute{e}t}} \arrow[d, "e"] & \\
SH(S_{\mathrm{r\acute{e}t}}) \arrow[ru] \arrow[r, "a"] \arrow[d] & SH^{S^1}(S)^{\mathrm{r\acute{e}t}}[\rho^{-1}] \arrow[d, "b"] & SH^{S^1}(S)[\rho^{-1}] \arrow[l] \arrow[d, "b'"] \\
SH(S)^{\mathrm{r\acute{e}t}} \arrow[r, "c"'] & SH(S)^{\mathrm{r\acute{e}t}}[\rho^{-1}] & SH(S)[\rho^{-1}] \arrow[l, "d"]
\end{tikzcd}$$

Furthermore, all the pseudo-functors ($S \mapsto SH(S)[\rho^{-1}]$ etc.) in the above diagram verify the full six functors formalism and continuity.

Rk: Bachmann states that the same results (except possibly $\nearrow \downarrow e$ which is not mentioned) are true for $D_{\mathbb{A}^1}$ (the stable $\mathbb{A}^1$-derived categories), the proofs being "exactly the same".

Proof: The last statement of the theorem follows from the first statement and Prop 28 (see Linda's talk).

The fact that b and b' are equivalences follow from Cor 27 (see Linda's talk).

″ c is an equivalence follows from Prop 29 (see Prop B in Linda's talk).

″ e is an equivalence ″ Prop C in Linda's talk.

″ d ″ II of this talk.

″ $b \circ a$ ″ III of this talk.

One concludes from the commutativity of the diagram above and the following lemma:

Lemma: If $g : B \to C$ is an equivalence and $f : A \to B$ is a functor such that $g \circ f$ is an equivalence then f is an equivalence.

Proof: $\forall a_1, a_2 \in A$ $\mathrm{Hom}_B(f(a_1), f(a_2)) \overset{g \text{ is fully faithful}}{\simeq} \mathrm{Hom}_C(g(f(a_1)), g(f(a_2))) \overset{g\circ f \text{ is fully faithful}}{\simeq} \mathrm{Hom}_A(a_1, a_2)$ hence f is fully faithful.

Let $b_1 \in B$. Since g is an equivalence, it has an "inverse" $h : C \to B$ which is essentially surjective so that: $\exists c_1 \in C$ $h(c_1) \simeq b_1$. Since $g \circ f$ is ess. surj: $\exists a_1 \in A$ $g(f(a_1)) \overset{\beta}{\simeq} c_1$.

Since g and h are "inverses" of each other, there is a natural isomorphism $\varepsilon : \mathrm{Id}_B \simeq h \circ g$.

$f(a_1) \overset{\varepsilon(f(a_1))}{\simeq} h(g(f(a_1))) \overset{h(\beta)}{\simeq} h(c_1) \simeq b_1$ so that f is ess. surj. Hence, f is an equivalence.

Ⅱ The canonical functor $d: SH(S)[\rho^{-1}] \to SH(S)^{\lambda ab}[\rho^{-1}]$ is an equivalence

__Def:__ The Postnikov tower of an $\mathbb{A}^1$-connected pointed motivic space X is the following comm. diag.:

$$\begin{tikzcd} \cdots \arrow[r, "f_{n+1}"] & \tau_{\leq n} X \arrow[r, "f_n"] & \tau_{\leq n-1} X \arrow[r, "f_{n-1}"] & \cdots \arrow[r, "f_1"] & \tau_{\leq 0} X \\ & & X \arrow[ul, "p_n"] \arrow[u, "p_{n-1}"] \arrow[urr, bend right, "p_0"'] & & \end{tikzcd}$$

(f_n is an $\mathbb{A}^1$-fibration of fibre $K(\pi_n^{(\mathbb{A}^1)}(X), n)$)
(hom)
↑ Eilenberg-MacLane space ($\pi_n^{(\mathbb{A}^1)} = \pi_n^{(\mathbb{A}^1)}(X)$, other=0)

with: $\begin{cases} \forall\, i > n \quad \pi_i^{(\mathbb{A}^1)}(\tau_{\leq n}) = 0 \\ \forall\, i \leq n \quad \pi_i^{(\mathbb{A}^1)}(p_n) : \pi_i^{(\mathbb{A}^1)}(X) \to \pi_i^{(\mathbb{A}^1)}(\tau_{\leq n} X) \text{ is an iso.} \\ X \xrightarrow{\text{can.}} \operatorname{holim}_n \tau_{\leq n} X \text{ is an } \mathbb{A}^1\text{-weak equivalence} \end{cases}$

$X_1 = X$ and: $\forall\, i \geq 2$ $X_i = \operatorname{hofib}(p_{i-2})$ (hom) fibre of p_{i-2} / $\pi_j(X_i) = 0$ if $j \leq i-2$, $\pi_j(X_i) \simeq \pi_j(X)$ otherwise

(Whitehead tower)

$\forall\, i \geq 1$ $g_i : X_{i+1} \to X_i$

$\forall\, i \geq 1$ $Y_i = (\text{ho})\operatorname{fib}(g_i)$ ($Y_1 \simeq \{*\}$)

$\forall\, i \geq 2$ $Y_i \simeq K(\pi_{i-1}^{(\mathbb{A}^1)}(X), i-2)$
($\pi_{i-2}(Y_i) \simeq \pi_{i-1}(X)$, other=0)

__Def:__ A Postnikov system for an object X in a triangulated category $\mathcal{E}$ is a diagram $\mathcal{P}$:

(see Fabio Tanania's "A Bille-type spectral sequence for motivic cohomology")
↑ [Tanania 2024]

$$\begin{tikzcd} \cdots \arrow[r] & X_n \arrow[r] \arrow[d] & X_{n-1} \arrow[r] \arrow[d] & \cdots \arrow[r] & X_1 = X \arrow[d] \\ \cdots & Y_n \arrow[ul, "{[1]}"] & Y_{n-1} \arrow[ul, "{[1]}"] & & Y_1 \arrow[ul, "{[1]}"] \end{tikzcd}$$

with all its triangles being distinguished triangles in $\mathcal{E}$. (See above for the P. system from a P. tower)

__Def:__ An exact couple in an abelian category $\mathcal{A}$ is a triangle $\begin{tikzcd} D \arrow[rr, "i"] & & D \arrow[dl, "j"] \\ & E \arrow[ul, "k"] & \end{tikzcd}$ which is exact at each vertex.

__Def:__ A cohomological functor $H: \underset{\text{triang cat}}{\mathcal{E}} \to \underset{\text{abelian cat}}{\mathcal{A}}$ is an additive contravariant functor which sends all distinguished triangles in $\mathcal{E}$ to long exact sequences in $\mathcal{A}$. specifically, $\cdots \to Z[-1] \to \underbrace{X \to Y \to Z \to X[1]}_{\text{dist. triangle}} \to Y[1] \to Z[1] \to \cdots$ is exact

__Def:__ The exact couple in $\mathcal{A}$ associated to the Postnikov system $\mathcal{P}$ in $\mathcal{E}$ and the cohomological functor $H: \mathcal{E} \to \mathcal{A}$ is: $\begin{tikzcd} D \arrow[rr, "i"] & & D \arrow[dl, "j"] \\ & E \arrow[ul, "k"] & \end{tikzcd}$ with, $\forall\, s, t \in \mathbb{Z}$ $\begin{cases} D^{s,t} := H^t(X_s) := H(X_s[-t]) \\ E^{s,t} := H^t(Y_s) := H(Y_s[-t]) \end{cases}$ ← Adams grading $\left(\begin{matrix} X_s = X_1 \\ Y_s = Y_1 \end{matrix} \text{ if } s \leq 1\right)$ and:

$\begin{cases} i : D^{s,t} \to D^{s+1,t} & \text{induced by } X_{s+1} \to X_s \\ j : D^{s,t} \to E^{s-1,t+1} & \text{induced by } Y_{s-1} \to X_s[1] \\ k : E^{s,t} \to D^{s,t} & \text{induced by } X_s \to Y_s \end{cases}$

$(p, q) = (-s, t+s)$ (usual convention in alg. geom)
$D^{p,q} = H^{p+q}(X_{-p})$, $E^{p,q} = H^{p+q}(Y_{-p})$
$i : D^{p,q} \to D^{p-1,q+1}$, $j : D^{p,q} \to E^{p+1,q}$,
$k : E^{p,q} \to D^{p,q}$.

__Prop:__ This is an exact couple in $\mathcal{A}$.

__Def:__ The differential $d : E \to E$ associated to the exact couple $\begin{tikzcd} D \arrow[rr, "i"] & & D \arrow[dl, "j"] \\ & E \arrow[ul, "k"] & \end{tikzcd}$ is $d = j \circ k$. __Rk:__ $d \circ d = 0$.

__Def:__ The derived couple $\begin{tikzcd} D' \arrow[rr, "i'"] & & D' \arrow[dl, "j'"] \\ & E' \arrow[ul, "k'"] & \end{tikzcd}$ associated to the exact couple ↑ in $\mathcal{A}$ is $D' := \operatorname{Im}(i)$, $E' := \operatorname{Ker}(d)/\operatorname{Im}(d)$, $i' = i|_{D'}^{D'}$, $j' : i(x) \longmapsto j(x)$ and $k' : [y] \mapsto k(y)$.

__Prop:__ This is an exact couple in $\mathcal{A}$. __Rk:__ Sequence of derived couples! $\begin{tikzcd} D_r \arrow[rr, "i_r"] & & D_r \arrow[dl, "j_r"] \\ & E_r \arrow[ul, "k_r"] & \end{tikzcd}$ derived couple $\Rightarrow \begin{cases} d_r \text{ is the differential of } E_r \text{ (the exact couple)} \\ E_{r+1} \text{ is the homology of } E_r \\ \text{(being part of the derived couple)} \end{cases}$

__Def:__ The spectral sequence associated to the exact couple $\begin{tikzcd} D_1 \arrow[rr, "i_1"] & & D_1 \arrow[dl, "j_1"] \\ & E_1 \arrow[ul, "k_1"] & \end{tikzcd}$ is $\{(E_r, d_r)\}_{r \geq 1}$ defined by:

Def: A bigraded spectral sequence $\{(E_r^{s,t}, d_r^{s,t})_{s,t\in\mathbb{Z}}\}_{r\geq 1}$ converges to $(E_\infty^{s,t})_{s,t\in\mathbb{Z}}$ if:

$\forall s,t\in\mathbb{Z}\ \exists r(s,t)\in\mathbb{N}\ \forall r\geq r(s,t)\ E_r^{s,t}\simeq E_\infty^{s,t}$.

Def: An increasing filtration $\mathcal{F}$ of an object G in $\mathcal{A}$ / a graded object $(G^t)_{t\in\mathbb{Z}}$ is: $\cdots\hookrightarrow F^n\hookrightarrow F^{n+1}\hookrightarrow\cdots$ ($\hookrightarrow$: monomorphism), $F^n\subseteq G$, $F^{n+1}\subseteq G$ / $\cdots\hookrightarrow F^{n,t}\hookrightarrow F^{n+1,t}\hookrightarrow\cdots$, $F^{n,t}\subseteq G^t$, $F^{n+1,t}\subseteq G^t$.

It is said to be:

- exhaustive if $G\simeq\operatorname{colim}_n F^n$;
- Hausdorff if $\lim_m F^m=0$;
- complete if $\lim^1_m F^m=0$ (where $\lim^1=R\lim$, the (first) derived functor of lim).

Def. A bigraded spectral sequence $\{(E_r^{s,t}, d_r^{s,t})_{s,t\in\mathbb{Z}}\}_{r\geq 1}$ converges weakly (resp. converges, converges strongly) to $(G^t)_{t\in\mathbb{Z}}$ if there is an exhaustive (resp. exhaustive and Hausdorff, exhaustive and Hausdorff and complete) increasing filtration $\mathcal{F}$ of $(G^t)_{t\in\mathbb{Z}}$ such that $\{(E_r^{s,t}, d_r^{s,t})\}$ converges to $(F^{s,t}/F^{s-1,t})_{s,t\in\mathbb{Z}}$.
(Goal: show there is $E_\infty^{s,t}$ and that $E_\infty^{s,t}\simeq E_0^{s,t}$; + exhaustive: link with G^t) $E_0^{s,t}$ (the "target")

Def: The increasing filtration ass. to the Postnikov system P in $\mathcal{E}$ and the cohom. functor $H:\mathcal{E}\to\mathcal{A}$ is:

$\cdots\hookrightarrow F^{s-1,t}\hookrightarrow F^{s,t}\hookrightarrow\cdots$ with $F^{s,t}:=\mathrm{Ker}(H^t(X)\to H^t(X_s))$ ($X_s\to X_{s-1}\to\cdots\to X_1=X$ in P ↘ $H^t(X)\to\cdots\to H^t(X_{s-1})\to H^t(X_s)$).
$F^{s-1,t}\subseteq H^t(X)=H(X[-t])$, $F^{s,t}\subseteq H^t(X)$

Prop: It is a Hausdorff and complete increasing filtration of $(H^t(X))_{t\in\mathbb{Z}}$.

Thm ([Boardman '99, Thm 6.1] "Conditionally convergent spectral sequences" in "Homotopy invariant alg. str." (AMS) cited as [Tanania 2024, Thm 3.7]): Let P be a Postnikov system in $\mathcal{E}$ and $H:\mathcal{E}\to\mathcal{A}$ be a cohomological functor such that $\operatorname{colim}_s H^*(X_s)=0$. The spectral sequence ass. to the exact couple ass. to P and H converges strongly to $H^*(X)$ (i.e. to $(H^t(X))_{t\in\mathbb{Z}}$).

Rk. I think the increasing filtration ass. to P and H works↑.

↙(the real-étale top. is finer than the Nisnevich top.)

Prop 31. Let k be a field of char. 0. The localization functor $L: SH(k)[\rho^{-1}]\to SH(k)^{ret}[\rho^{-1}]$ is an equiv.
↙seen as the ρ-local objects of $SH(k)$

Proof: It is enough to show that each $E\in SH(k)[\rho^{-1}]$ is ret-local, i.e.:

$\forall X\in Sm_k\ \forall U_\bullet=(U_n)_{n\in\mathbb{N}_0}\to X$ ret-hypercover (/local acyclic fibration, see "Hypercovers and simplicial presheaves" by Dugger, Hollander, Isaksen)
↖($U_\bullet$ is a simplicial scheme)

$E(X):=[\Sigma^\infty X_+, E]_{SH(k)}\to[\Sigma^\infty(U_\bullet)_+, E]_{SH(k)}=:E(U_\bullet)$ (induced by) is an isomorphism.

The Postnikov tower of E and the cohomological functor H_{Nis} give rise, for each $i\in\mathbb{Z}$, to a spectral sequence of E_2'-page $H^p_{Nis}(X,\underline{\pi}_{-q}(E)_{-i})$ and of target $[\Sigma^\infty X_+\wedge\mathbb{G}_m^{\wedge i}, E\wedge S^{p+q}]_{SH(k)}$ $(E^{p+q-i,-i}(X))$ $=:E[p+q]$
$\simeq[\Sigma^\infty X_+,\underline{\pi}_{-q}(E)_{-i}\wedge S^p]_{SH(k)}$ $\supseteq F^{p,q}/F^{p-1,q}$ (before we know it is $E_\infty^{p,q}$)

and to a morphism of spectral sequences from this first spectral seq. to this second spectral seq. ($f_r: E_r'\to E_r''$ s.t. $f_r\circ d_r'=d_r''\circ f_r$ and f_{r+1} is induced by f_r ($E_{r+1}^{(\prime)}$ = homology of $E_r^{(\prime)}$))

and to a spectral sequence of E_2''-page:

$[U_\bullet,\underline{\pi}_{-q}(E)_{-i}\wedge S^p]_{SH(k)}$ and of target $[U_\bullet\wedge\mathbb{G}_m^{\wedge i}, E\wedge S^{p+q}]_{SH(k)}$ $(=:E^{p+q-i,-i}(U_\bullet))$.
and ret-hypercover (as well as a m. of spectral seq. from the first to the second)

We also have the simplicial spectral sequences of E_1-pages: ("homotopy colimit spectral seq." in [Boardman] because the index $\approx$ not $U_\bullet$)
$[\Sigma^\infty(U_n)_+,\underline{\pi}_{-q}(E)_{-i}\wedge S^p]_{SH(k)}$ and of targets $[\Sigma^\infty(U_\bullet)_+,\underline{\pi}_{-q}(E)_{-i}\wedge S^{p+n}]_{SH(k)}$
$H^p_{ret}(U_n,\underline{\pi}_{-q}(E)_{-i})$ $H^{p+n}_{ret}(X,\underline{\pi}_{-q}(E)_{-i})$, ← (implicit in [Boardman])

Note that $[U_n, \underline{\Pi}_{-q}(E)_{-i} \overbrace{\wedge S^p]_{SH(k)}}^{[\rho]} \simeq H^p_{Nis}(U_n, \underline{\Pi}_{-q}(E)_{-i})$ and that, since E is ρ-local, the morphism $\underline{\Pi}_{-q}(E)_{-i} = [S^{-q} \wedge \mathbb{G}_m^{\wedge i}, E]_{SH(k)} \to [S^{-q} \wedge \mathbb{G}_m^{\wedge(i-1)}, E]_{SH(k)} = \underline{\Pi}_{-q}(E)_{-i+1}$ induced by ρ is an isomorphism so that, by Theorem 25 (see Jan's talk), $\underline{\Pi}_{-q}(E)$ is a completed ret-sheaf so that, by Prop 19.2.1 (& Thm 19.2) in [Scheiderer 1994], $H^p_{ret}(U_n, \underline{\Pi}_{-q}(E)_{-i}) \simeq H^p_{Zar}(U_n, \underline{\Pi}_{-q}(E)_{-i}) \simeq H^p_{Nis}(\underset{Sm_k}{U_n}, \underset{\substack{\uparrow \\ \text{strongly } \mathbb{A}^1\text{-inv. sheaf of abelian groups}}}{\underline{\Pi}_{-q}(E)_{-i}})$ (by Ex 5.43 in Morel's 2012 book "$\mathbb{A}^1$-Alg. top. over a field") so that finally:

(*) $\quad [U_n, \underline{\Pi}_{-q}(E)_{-i} \wedge S^p]_{SH(k)} \simeq H^p_{ret}(U_n, \underline{\Pi}_{-q}(E)_{-i})$ (and this iso. is the morphism which is part of the m. of spectral seq. from the simplicial s.s. to the ret-hyp. s.s. ; all the following f_n are iso. (since f_n is induced by f_1))

The dimension of X being finite (as it is a (smooth) finite-type k-scheme), its cohomological dimension is finite and the simplicial (resp. ret-hypercover) spectral s. converges strongly to $[U_\bullet, \underline{\Pi}_{-q}(E)_{-i} \wedge S^{p+n}]_{SH(k)}$ (resp. to $H^{p+n}_{ret}(X, \underline{\Pi}_{-q}(E)_{-i})$) so that, by (*) above, $[U_\bullet, \underline{\Pi}_{-q}(E)_{-i} \wedge S^{p+n}]_{SH(k)} \simeq H^{p+n}_{ret}(X, \underline{\Pi}_{-q}(E)_{-i})$.

As above, $H^{p+n}_{ret}(X, \underline{\Pi}_{-q}(E)_{-i}) \simeq H^{p+n}_{Nis}(X, \underline{\Pi}_{-q}(E)_{-i})$ so that in particular we have an isomorphism between the E_2-pages of the first two spectral sequences mentioned above. This isomorphism is in fact the morphism which is part of the morphism of spectral sequences from the first s.s mentioned above to the second s.s. mentioned above, so that all the following morphisms f_n are isomorphisms. Since those spectral sequences converge strongly to their targets (for cohomological dimension reasons), we get an isomorphism $f_\infty : E^{p+q-i,-i}(X) \xrightarrow{\sim} E^{p+q-i,-i}(U_\bullet)$ and in particular (for $p=q=i=0$) an iso $f_\infty : E(X) \to E(U_\bullet)$ which is in fact the morphism induced by the ret-hypercover $U_\bullet \to X$.

Ex 32: Let k be a field. The localization functor $L : SH(k)[\rho^{-1}] \to SH(k)^{ret}[\rho^{-1}]$ is an equivalence.

Proof. Prop 31 gives the result for $\mathrm{char}(k)=0$. Suppose $\mathrm{char}(k)=p>0$. ; we will show that ρ is exactly nilpotent in $SH(k)$. (□) Wlog (by base change), suppose $k=\mathbb{F}_p$.

It is enough to show (□) since in positive char. $SH(k)^{ret}=0$ (since $R(k)$ is empty).

Let us show that $\operatorname{colim}_n K_n^{MW}(\mathbb{F}_p) = 0$ (where $K_n^{MW}(\mathbb{F}_p) \xrightarrow{\times \rho} K_{n+1}^{MW}(\mathbb{F}_p)$).

This will follow from $\operatorname{colim}_n I^n(\mathbb{F}_p) = 0$ (where $I^n(\mathbb{F}_p) \xrightarrow{\times 2} I^{n+1}(\mathbb{F}_p)$) as in Ex 19's proof (see Jan's talk).

This follows from the fact that $I^2(\mathbb{F}_p)=0$ ($\Rightarrow I^n(\mathbb{F}_p)=0$ for all $n \geq 2$). Indeed:

- If $p=2$ then $W(\mathbb{F}_p) \simeq \mathbb{Z}/2\mathbb{Z}$ via the rank modulo 2 hence $I(\mathbb{F}_p)=0$ hence $I^2(\mathbb{F}_p)=0$.
- If $p \equiv 3[4]$ then $W(\mathbb{F}_p) \simeq \mathbb{Z}/4\mathbb{Z}$ via the signature hence $I(\mathbb{F}_p)=(2)$ in $\mathbb{Z}/4\mathbb{Z}$ hence $I^2(\mathbb{F}_p)=0$.
- If $p \equiv 1[4]$ then $W(\mathbb{F}_p) \simeq \mathbb{Z}/2\mathbb{Z}[\mathbb{Z}/2\mathbb{Z}]$ (← binary group ring) via the signature couple hence $I(\mathbb{F}_p)=(1+\underset{\substack{\uparrow \\ \text{non-square}}}{\langle s\rangle})$ hence $I^2(\mathbb{F}_p)=0$ (since the non-square product $(1\times 1 = 1, \langle s\rangle \times \langle s\rangle = 1, 1 \times \langle s\rangle = \langle s\rangle)$ $(1+\langle s\rangle)(1+\langle s\rangle) = 1+2\langle s\rangle+1 = 0$).

Ex 33: The can. localization functor $L : SH(S)[\rho^{-1}] \to SH(S)^{ret}[\rho^{-1}]$ is an equivalence.

Proof. It is enough to show that: $\forall X \in Sm_S$ $\forall (\Sigma^\infty(U_n)_+)_{n \in \mathbb{N}_0} \to X$ ret-hypercover $\alpha : \operatorname{hocolim}_n \Sigma^\infty(U_n)_+ \to \Sigma^\infty X_+$ is an equivalence.

By Ex 14 (see Ehivantan's talk), which holds for $SH(S)[\rho^{-1}]$ by Prop 28 (see Zenda's talk), it is enough to show that: $\forall f : \operatorname{Spec}(\overset{\text{field}}{k}) \to S$ $f^*\alpha$ is an equivalence. Since $f^*\alpha \overset{\leftarrow f^* \text{ left adjoint}}{\simeq} \operatorname{hocolim}_n \Sigma^\infty(\overset{\leftarrow \text{ ret-hypercover of } f^*X}{f^*U_n})_+ \to \Sigma^\infty(f^*X)_+$, this follows from Ex 32

Ⓘⓘⓘ The canonical functor $b \circ a : SH(S_{r\acute{e}t}) \to SH(S)^{r\acute{e}t}[\rho^{-1}]$ is an equivalence

$S_{r\acute{e}t}$: small real-étale site for S (étale $\overset{\text{(sep, f-t)}}{\vee}$ schemes over S with the real-étale topology)

$e \big\downarrow$ $Sm(S)_{r\acute{e}t}$: smooth (sep, finite-type) schemes over S with the real-étale topology

In Marc's second talk, we saw:

$$e^! : Psh(S_{r\acute{e}t}) \overset{\text{adj.}}{\leftrightarrows} Psh(Sm(S)_{r\acute{e}t}) : r$$

$$\underset{\substack{\uparrow \\ \text{fully faithful}}}{e} : Shv(S_{r\acute{e}t}) \overset{\text{geom. morphism}}{\leftrightarrows} Shv(Sm(S)_{r\acute{e}t}) : \underset{\substack{\uparrow \\ \text{preserves colimits}}}{r} \quad \text{(Lemma 5)}$$

$\underset{\substack{\uparrow \\ \text{derived functor}}}{Le} : SH(S_{r\acute{e}t}) \to SH(Sm(S)_{r\acute{e}t})$ is fully faithful and t-exact (Ex 6)

$$\begin{cases} SH^{S^1}(S)^{r\acute{e}t} \text{ is the } \mathbb{A}^1\text{-localization of } SH(Sm(S)_{r\acute{e}t}) \\ SH^{S^1}(S)^{r\acute{e}t}[\rho^{-1}] \text{ is the } (\mathbb{A}^1,\rho)\text{-localization of } SH(Sm(S)_{r\acute{e}t}) \rightsquigarrow L_{\mathbb{A}^1,\rho} : SH(Sm(S)_{r\acute{e}t}) \to SH^{S^1}(S)^{r\acute{e}t}[\rho^{-1}] \end{cases}$$

$a = L_{\mathbb{A}^1,\rho} \circ Le$

Prop 34: The canonical functor $b \circ a : SH(S_{r\acute{e}t}) \to SH(S)^{r\acute{e}t}[\rho^{-1}]$ is an equivalence.

Proof: $\overset{\text{see above}}{\downarrow}$ Le is fully faithful and t-exact (so that $\forall E \in SH(S_{r\acute{e}t})\ \forall i \in \mathbb{N}_0\ \pi_i(LeE) \simeq e(\pi_i(E))$).

+ by Thm 8 (see Marc's second talk), for all $F \in Shv(S_{r\acute{e}t})$, $H^p_{r\acute{e}t}(S \times \mathbb{A}^1, F) \simeq H^p_{r\acute{e}t}(S, F)$ and $H^p_{r\acute{e}t}(S_+ \wedge \mathbb{G}_m, F) \simeq H^p_{r\acute{e}t}(S, F)$

+ descent spectral sequence (as in Ⓘⓘ)

⇓ the image of Le consists of $\mathbb{A}^1$-local and ρ-local objects of $SH(Sm(S)_{r\acute{e}t})$ and $L_{\mathbb{A}^1,\rho} \circ Le$ is fully faithful
by Ex 27 ↘ (see Zander's talk)
i.e. a is fully faithful. Hence, $b \circ a$ is fully faithful (b being fully faithful as it is an equivalence)

Let us now show that $b \circ a$ is essentially surjective.

By Prop. 4.2.13 in "Triangulated categories of mixed motives" (Cisinski–Déglise), the category $SH(S)^{r\acute{e}t}[\rho^{-1}]$ is the smallest thick (i.e. stable by direct factors) triangulated full subcategory of $SH(S)^{r\acute{e}t}$ which contains objects of the form $p_*(\mathbb{1}_T)$ with $p : T \to S$ projective. (*)

Since e, hence $b \circ a$, has a right adjoint, $b \circ a$ commutes with arbitrary sums and thus identifies $SH(S_{r\acute{e}t})$ with a localizing subcategory (i.e. a triangulated full subcategory which is stable by small sums) of $SH(S)^{r\acute{e}t}[\rho^{-1}]$. Combining this with (*), it suffices to show that $b \circ a$ commutes with p_* for every projective morphism $p : T \to S$. The proof of this statement is "exactly the same" as the proof of Prop. 4.4.3 in "Étale motives" (Cisinski–Déglise), replacing f with p and $\rho_!$ with $b \circ a$. This proof uses the following properties (which should follow from those of Le): monoidality (1), commutation with pullbacks $\overset{(2)}{\vee}$ and commutation with $g_\#$ for all étale g (3) (as well as the fact that p is proper since it is projective).

The idea of the proof is as follows:

- By the Yoneda lemma, it is enough to show that for each object K in $SH(S_{r\acute{e}t})$ and each object M in $SH(S)^{r\acute{e}t}[\rho$

  $\varphi_p : \mathrm{Hom}(M, \mathcal{B}(a(p_*(K)))) \longrightarrow \mathrm{Hom}(M, p_*(\mathcal{B}(a(K))))$ is an isomorphism (with $p: T \to S$ projective).

  can be replaced with $q_\#(\mathbb{1}_W)[n]$, $n \in \mathbb{Z}$ where $q: W \to S$ is smooth (or something similar) ← uses (3)

  induced by $\mathcal{B}\circ a\circ p_* \xrightarrow{adj} p_*\circ p^*\circ\mathcal{B}\circ a\circ p_* \xrightarrow{(2)} p_*\circ\mathcal{B}\circ a\circ p^*\circ p_* \xrightarrow{adj} p_*\circ\mathcal{B}\circ a$

- Every cartesian square

$$\begin{tikzcd} Y \arrow[r, "q'"] \arrow[d, "p'"'] \arrow[dr, phantom, "\lrcorner", very near start] & T \arrow[d, "p\ \mathrm{proj}"] \\ W \arrow[r, "q"'] & S \end{tikzcd}$$

  $(\Rightarrow \text{proper})$ gives rise to an isomorphism $q^* \circ p_* \xrightarrow{\sim} p'_* \circ (q')^*$ in $SH(-_{r\acute{e}t})$ (by Thm 9, see Marc's second talk) and an iso. $q^* \circ p_* \xrightarrow{\sim} p'_* \circ (q')^*$ in $SH(-)^{r\acute{e}t}[\rho^{-1}]$ ← proper base change (by the six functors formalism in $SH(-)[\rho^{-1}]$ (see Prop. 28 in Linda's talk) and the equivalence of (see $\textcircled{II}$)) so that with (2) we get the following commutative diagram whose vertical arrows are isomorphisms:

$$\begin{tikzcd} q^*\mathcal{B}a\,p_* \arrow[r, "\varphi_p"] \arrow[d, "(2)"'] & q^*p_*\mathcal{B}a \arrow[d, "\text{proper base change in } SH(-)^{r\acute{e}t}[\rho^{-1}]"] \\ \mathcal{B}a\,q^*p_* \arrow[d, "\text{proper base change in } SH(-_{r\acute{e}t})"'] & p'_*(q')^*\mathcal{B}a \arrow[d, "(2)"] \\ \mathcal{B}a\,p'_*(q')^* \arrow[r, "\varphi_{p'}"'] & p'_*\mathcal{B}a\,(q')^* \end{tikzcd}$$

  This, together with the adjunction $q_\# \dashv q^*$ for $q: W \to S$ smooth (and by replacing K by $(p')^*(K)\boxtimes$ or something similar), means that it suffices to show that $\varphi_p : \mathrm{Hom}(\mathbb{1}_S, \mathcal{B}(a(p_*(K)))) \longrightarrow \mathrm{Hom}(\mathbb{1}_S, p_*(\mathcal{B}(a(K))))$ is an isomorphism (with $p: T \to S$ projective and K in $SH(S_{r\acute{e}t})$).

- For every $p: T \to S$ projective and K in $SH(S_{r\acute{e}t})$, $\varphi_p : \mathrm{Hom}(\mathbb{1}_S, \mathcal{B}(a(p_*(K)))) \to \mathrm{Hom}(\mathbb{1}_S, p_*(\mathcal{B}(a(K))))$ is the composite:

$$\begin{tikzcd} \mathrm{Hom}(\mathbb{1}_S, \mathcal{B}(a(p_*(K)))) \arrow[d, "\wr"', "(1)"] \\ \mathrm{Hom}(\mathcal{B}(a(\mathbb{1}_S)), \mathcal{B}(a(p_*(K)))) \arrow[d, "\wr"', "\mathcal{B}\circ a \text{ is fully faithful}"] \\ \mathrm{Hom}(\mathbb{1}_S, p_*(K)) \arrow[d, "\wr"', "p^* \dashv p_*"] \\ \mathrm{Hom}(p^*(\mathbb{1}_S), K) \arrow[d, "\wr"', "\mathcal{B}\circ a \text{ is fully faithful}"] \\ \mathrm{Hom}(\mathcal{B}(a(p^*(\mathbb{1}_S))), \mathcal{B}(a(K))) \arrow[d, "\wr"', "(2)"] \\ \mathrm{Hom}(p^*(\mathcal{B}(a(\mathbb{1}_S))), \mathcal{B}(a(K))) \arrow[d, "\wr"', "p^* \dashv p_*"] \\ \mathrm{Hom}(\mathcal{B}(a(\mathbb{1}_S)), p_*(\mathcal{B}(a(K)))) \arrow[d, "\wr"', "(1)"] \\ \mathrm{Hom}(\mathbb{1}_S, p_*(\mathcal{B}(a(K)))) \end{tikzcd}$$

  and these are all isomorphisms hence φ_p is an isomorphism.