INTEGRALS OF ψ-CLASSES ON TWISTED DOUBLE RAMIFICATION CYCLES AND SPACES OF DIFFERENTIALS

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ABSTRACT. We prove a closed formula for the integral of a power of a single ψ-class on strata of k-differentials. In many cases, these integrals correspond to intersection numbers on twisted double ramification cycles.

Then we conjecture an expression of a refinement of double ramification cycles according to the parity of spin structures. Assuming that this conjecture is valid, we also compute the integral of a single ψ-class on the even and odd component of strata of k-differentials.

As an application of these results we give a closed formula for the Euler characteristic of components of minimal strata of abelian differentials.

CONTENTS

1. INTRODUCTION

The Hodge bundle over the moduli space of curves is the moduli space of abelian differentials, i.e. Riemann surfaces endowed with an holomorphic 1-form. The complement of the zero section admits a stratification according to the orders of the zeros of the one-form. These strata may be studied from different perspectives.

These strata appear naturally in many areas of mathematics, such as the study of flat surfaces, Teichmüller dynamics and associated counting problems (see [EM18], [EMM15], [Fil16], [EFW18], [EMMW20] to only cite a few of many works in these directions).

Since these spaces can be described as algebraic objects, we may use intersection theory to understand their numerical invariants and dynamics. For instance, using a smooth compactification of the moduli spaces of abelian differentials ([BCG$^+$19b], [CMZ19]), it was possible to apply intersection theory to compute the Masur-Veech volumes and the sum of the Lyapunov exponents of the strata ([Sau18], [CMSZ20]), as well as their Euler characteristic ([CMZ20a]). More generally one may consider

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the moduli spaces of holomorphic k-forms, and once again numerical invariants of these spaces may be computed by intersection theory (see [CMS19], [DGZZ21] and [Sau20]).

Another interesting connection to enumerative geometry is the relation between strata of k-differentials and twisted double ramification cycles (see [FP18, Sch18, HS21, BHP$^+$20]). Classical double ramification cycles were introduced to study Gromov-Witten invariants via degeneration techniques ([Li02]). For instance, the Gromov-Witten theory of curves may be reduced to integrals of ψ-classes on double ramification cycles ([OP06], [BSSZ15]).

The aim of the present work is to generalize the results of [BSSZ15] and compute integrals of the top power of a single ψ-class on strata of k-differentials and on twisted double ramification cycles. The results we present allow both to simplify the computations appearing in [Sau20] about volumes of strata of flat surfaces with rational singularities and to give an explicit formula for the orbifold Euler characteristic of minimal strata of abelian differentials based on results from [CMZ20a]. This last application allows us to formulate a conjecture about the asymptotic growth of the orbifold Euler characteristic of such strata for large values of the genus.

Moreover, if the zeros of a 1-form are even, one may associate a canonical spin structure to it (i.e. a square root of the cotangent bundle of the curve). This spin structure defines an invariant called the parity of the spin structure. Below, we formulate a conjecture on a spin analogue of double ramification cycles. We show that this conjecture allows to compute intersection numbers of powers of ψ-classes and the orbifold Euler characteristics of strata of differentials weighted by a sign depending on the parity of the spin structure.

We will now provide a more in depth summary of the results of this work.

1.1. **Strata of k-differentials.** Let g and n be non-negative integers satisfying $2g-2+n>0$. We denote by $\mathcal{M}_{g,n}$ and $\overline{\mathcal{M}}_{g,n}$ the moduli spaces of genus g, smooth and stable curves with n distinct markings, respectively.

Let k be an integer, and let $a=(a_1,\dots,a_n)\in\mathbb{Z}^n$ be a vector satisfying

$$|a| \coloneqq \sum_{i=1}^{n} a_i = k(2g-2+n).$$

We denote[1] by $\mathcal{M}_g(a)\subset\mathcal{M}_{g,n}$ the locus of marked curves $(C,x_1,\dots,x_n)$ such that

$$\omega_{\log}^{\otimes k} \simeq \mathcal{O}_C\left(a_1x_1+\ldots+a_nx_n\right). \tag{1}$$

Since $\mathcal{M}_g(-a)=\mathcal{M}_g(a)$, we can restrict our study to the case $k\geq 0$ from now on. For $k>0$ we call $\mathcal{M}_g(a)$ the *stratum of k-differentials of type a*, while if $k=0$ we call it the *Hurwitz scheme of type a*. To justify these names, observe that for $k>0$ the equality (1) of line bundles is equivalent to the existence of a meromorphic k-differential on C with zeros and poles at the points x_i of orders a_i-k, and this k-differential is unique up to scaling. On the other hand, for $k=0$ condition (1) is equivalent to the existence of a Hurwitz cover $C\to\mathbb{P}^1$ with the points x_i forming the preimages of $0,\infty$ and local ramification orders a_i (over 0) and $-a_i$ (over ∞).

[1] Note that we do not include k in the above notation as it is determined from a and g.

The space $\mathcal{M}_g(a)$ is a smooth subspace of $\mathcal{M}_{g,n}$ of dimension

$$\begin{cases} 3g-3+n, & \text{if } a=0,\ k=0 \\ (2g-2+n), & \text{if } a\in\mathbb{Z}^n_{>0}, \text{ and } k=1 \\ \text{mixed dimension}, & \text{if } a\in k\mathbb{Z}^n_{>0}, \text{ and } k>1 \\ (2g-3+n), & \text{otherwise} \end{cases}$$

If $k>0$, and $a\in k\mathbb{Z}_{>0}$, then the space $\mathcal{M}_g(a/k)$ of dimension $2g-2+n$ sits naturally inside $\mathcal{M}_g(a)$ as the open and closed subspace of k-differentials obtained as the kth power of a 1-differential. Meanwhile, all other components of $\mathcal{M}_g(a)$ are of dimension $2g-3+n$, thus explaining the mixed dimension in the third line. We denote by $\overline{\mathcal{M}}_g(a)$ the Zariski closure of $\mathcal{M}_g(a)$ in $\overline{\mathcal{M}}_{g,n}$.

1.2. **Double ramification cycles.** There exists a family of cycles, the *twisted*[2] *double ramification cycles*:

$$\mathrm{DR}_g : \mathbb{Z}^n \to A^g(\overline{\mathcal{M}}_{g,n}),$$

satisfying the following properties:

- we have $\mathrm{DR}_g(a)_{|\mathcal{M}_{g,n}} = [\mathcal{M}_g(a)] \in A^g(\mathcal{M}_{g,n},\mathbb{Q})$, if $a\notin k\mathbb{Z}_{>0}$,
- the function $\mathrm{DR}_g(a)$ is a cycle-valued polynomial of degree $2g$ in the entries of the vector a.

We refer the reader to Section 2.2 for a precise definition of these cycles. The purpose of the present paper is to compute the following polynomial defined by intersection theory:

$$\begin{aligned} \mathcal{A}_g : \mathbb{Z}^n &\to \mathbb{Q}, \\ a &\mapsto \int_{\overline{\mathcal{M}}_{g,n}} \mathrm{DR}_g(a)\cdot \psi_1^{2g-3+n}, \end{aligned} \tag{2}$$

where we recall that ψ_i is the Chern class of the cotangent line at the i-th marking for all $1\le i\le n$.

Theorem 1.1. *For all $g,n\ge 0$ satisfying $2g-2+n>0$ and $a\in\mathbb{Z}^n$, we have*

$$\mathcal{A}_g(a) = [z^{2g}]\ \exp\left(\frac{a_1 z\cdot \mathcal{S}'(kz)}{\mathcal{S}(kz)}\right)\frac{\prod_{i>1}\mathcal{S}(a_i z)}{\mathcal{S}(z)\mathcal{S}(kz)^{2g-1+n}}, \tag{3}$$

where $k=|a|/(2g-2+n)$, $\mathcal{S}(z)=\frac{\sinh(z/2)}{z/2}$, and $[z^{2g}](\cdot)$ stands for the coefficient of z^{2g} in the formal series.

This result extends Theorem 1 of [BSSZ15], which proved this formula in the case $|a|=k=0$. As in their paper, we prove the formula (3) by first giving a list of properties satisfied by the functions $\mathcal{A}_g$ which characterize them uniquely (see Lemma 4.1). Then we verify that the formula on the right-hand side of (3) indeed has all desired features and thus computes the value of $\mathcal{A}_g$. The analogous properties in [BSSZ15] for $k=0$ were proven using a definition of the double ramification cycle via the Gromov-Witten theory of the projective line. For $k>0$ no such definition is available. Instead we use properties of the formula for $\mathrm{DR}_g(a)$

[2] Here, the word *twisted* refers to the fact that the cycles compactify the condition $\omega_{\log}^{\otimes k}\cong \mathcal{O}_C(\sum_i a_i x_i)$ for arbitrary $k\in\mathbb{Z}$ instead of the more restrictive case $k=0$ appearing e.g. in the paper [BSSZ15]. In particular, we mention the word in the title of our paper to distinguish our results from theirs. That having been said, we will, in the interest of brevity, mostly omit the word *twisted* in the following text.

in terms of tautological classes studied in [JPPZ17, PZ] and results about the geometry of the strata of k-differentials (combined with their connection to the double ramification cycles).

One of the reasons why the case $k > 0$ of Theorem 1.1 is particularly interesting is that the corresponding values of $\mathcal{A}_g(a)$ are related to intersection numbers on strata of k-differentials. Indeed, we have

$$\mathcal{A}_g(a) = \begin{cases} \int_{\overline{\mathcal{M}}_{g,n}} [\overline{\mathcal{M}}_g(a)] \cdot \psi_1^{2g-3+n} & \text{if } k > 0 \text{ and } a_1 \notin k\mathbb{Z}_{>0}, \\ -a_1 \int_{\overline{\mathcal{M}}_{g,n}} [\overline{\mathcal{M}}_g(a)] \cdot \psi_1^{2g-2+n} & \text{if } k = 1 \text{ and } a \in \mathbb{Z}_{>0}^n. \end{cases} \tag{4}$$

This equality follows from a relationship between strata of k-differentials and double ramification cycles conjectured (see Conjecture A in [FP18, Sch18]) which was recently established in [HS21, BHP$^+$20] (see Section 2.2 for details). More generally, the intersection number of a power of a single ψ class on an arbitrary stratum of k-differentials may be recovered from the function $\mathcal{A}_g$ (see [Sau20], Section 2.2).

1.3. **Spin refinement.** If k is odd and a has only odd entries, then a marked curve $(C, x_1, \ldots, x_n) \in \mathcal{M}_g(a)$ carries a natural *spin structure* (i.e. a line bundle L satisfying $L^{\otimes 2} \simeq \omega_C$) defined by

$$L = \omega_C^{\otimes(-k+1)/2} \otimes \mathcal{O}_C\left(\frac{a_1 - k}{2} x_1 + \ldots + \frac{a_n - k}{2} x_n\right).$$

The parity of a spin structure $L \to C$, defined as the parity of $h^0(C, L)$, is invariant in a family of spin structures over smooth curves (see [Mum71]). In particular the space $\mathcal{M}_g(a)$ may be decomposed as the disjoint union

$$\mathcal{M}_g(a) = \mathcal{M}_g(a)^{\mathrm{odd}} \sqcup \mathcal{M}_g(a)^{\mathrm{even}}$$

of two (possibly themselves disconnected) subspaces defined by the parity of the above canonical spin structure.

Remark 1.2. The reader should be careful as there is another natural definition of the parity of a k-differential for $k > 1$. Indeed, one can define the parity of a k-differential as the parity of the associated *canonical cover* (this choice is made in [CG21] for example). This second definition is more natural from the point of view of flat surfaces (for example it is well defined for values of k even and positive). However, it does not agree with our definition of parity. For instance, the parity of a k-differential of genus 0 may be odd with this alternative definition while it is always even with the definition that we use in the text.

We will consider the following cycle

$$[\overline{\mathcal{M}}_g(a)]^{\mathrm{spin}} = [\overline{\mathcal{M}}_g(a)^{\mathrm{even}}] - [\overline{\mathcal{M}}_g(a)^{\mathrm{odd}}] \in A^*(\overline{\mathcal{M}}_{g,n}, \mathbb{Q}).$$

Moreover, if $a_1 \notin k\mathbb{Z}_{>0}$, we set:

$$\mathcal{A}_g^{\mathrm{spin}}(a) = \int_{\overline{\mathcal{M}}_{g,n}} [\overline{\mathcal{M}}_g(a)]^{\mathrm{spin}} \cdot \psi_1^{2g-3+n}$$

In the text, we will make the following assumption:

Assumption 1.3. There exists a family of cycles $\mathrm{DR}_g^{\mathrm{spin}} : \mathbb{Z}^n \to A^g(\overline{\mathcal{M}}_{g,n})$ such that:

(1) The function $\mathrm{DR}_g^{\mathrm{spin}}$ is a symmetric polynomial of degree $2g$;

(2) It holds

$$\pi^*\mathrm{DR}_g^{\mathrm{spin}}(a_1,\ldots,a_n) = \mathrm{DR}_g^{\mathrm{spin}}(a_1,\ldots,a_n,k)$$

where $\pi : \overline{\mathcal{M}}_{g,n+1} \to \overline{\mathcal{M}}_{g,n}$ is the forgetful morphism of the last marking and $k = (a_1 + \ldots a_n)/(2g-2+n)$;

(3) If a and k are odd, and P is a monomial in classes ψ_i for which a_i is negative or not divisible by k, we have

$$\int_{\overline{\mathcal{M}}_{g,n}} [\overline{\mathcal{M}}_g(a)]^{\mathrm{spin}} \cdot P = \int_{\overline{\mathcal{M}}_{g,n}} \mathrm{DR}_g^{\mathrm{spin}}(a) \cdot P;$$

in particular, when $a_1 \notin k\mathbb{Z}_{>0}$, we have

$$\mathcal{A}_g^{\mathrm{spin}}(a) \quad = \quad \int_{\overline{\mathcal{M}}_{g,n}} \mathrm{DR}_g^{\mathrm{spin}}(a) \cdot \psi_1^{2g-3+n},$$

and thus the function $\mathcal{A}_g^{\mathrm{spin}}(a)$ is a polynomial of degree $2g$;

(4) if a is positive and $k = 1$, the polynomial extension of $\mathcal{A}_g^{\mathrm{spin}}(a)$ satisfies

$$\mathcal{A}_g^{\mathrm{spin}}(a) \quad = \quad -a_1 \int_{\overline{\mathcal{M}}_{g,n}} [\overline{\mathcal{M}}_g(a)]^{\mathrm{spin}} \cdot \psi_1^{2g-2+n}.$$

Concerning the validity of the assumptions, we first note that all of them hold verbatim for the usual (i.e. non-spin) double ramification cycles and strata of differentials. Moreover, in Section 2.3 we present an explicit candidate for the function $\mathrm{DR}_g^{\mathrm{spin}}$. It satisfies parts (1) and (2) of the assumption by unpublished work [PZ] of Pixton and Zagier. Moreover, in Conjecture 2.5 we formulate a spin analogue of Conjecture A from [FP18, Sch18] which would imply the remaining parts of the assumption (see Proposition 2.6). We also describe the outline of a possible proof for this new conjecture.

Assuming the properties above, we show that the techniques used to prove Theorem 1.1 can be applied to give an explicit formula for $\mathcal{A}_g^{\mathrm{spin}}(a)$.

Theorem 1.4. *If parts (1)-(3) of Assumption 1.3 are valid, then we have:*

$$\mathcal{A}_g^{\mathrm{spin}}(a) \quad = \quad 2^{-g}[z^{2g}]\ \exp\left(\frac{a_1 z\cdot \mathcal{S}'(kz)}{\mathcal{S}(kz)}\right) \frac{\cosh(z/2)}{\mathcal{S}(z)} \frac{\prod_{i>1}\mathcal{S}(a_i z)}{\mathcal{S}(kz)^{2g-1+n}}. \tag{5}$$

1.4. **Applications.** In [Sau20], the second author showed that the volume of moduli spaces of flat surfaces with rational singularities, as well as the Masur-Veech volumes of strata of k-differentials (if all a_i are in $\mathbb{Z}_{>0} \setminus k\mathbb{Z}_{>0}$) may be computed. The input needed to calculate these volumes are the functions $\mathcal{A}_g$. Therefore the present work simplifies these computations by making the input explicit.

In Section 6 below we present a second application of this formula. The first author with M. Möller and J. Zachhuber, provided an expression for the orbifold Euler characteristic of strata of abelian differentials in terms of intersection numbers on these strata (see [CMZ20a]). We use this result together with Theorem 1.1 to obtain a closed formula for the orbifold Euler characteristic of the minimal strata $\mathcal{M}_g(2g-1)$ for $g \geq 1$.

In order to state this formula, we introduce the following formal series in $\mathbb{Q}(y)[\![z]\!]$:

$$\mathcal{H}(y,z) = \sum_{g\geq 1} (2g-1)!\, b_g \left(\frac{z\mathcal{S}(z)}{y}\right)^{2g} \frac{\mathcal{S}((2g-1)z)}{\mathcal{S}(z)},$$

where the numbers b_g are defined by $\mathcal{S}(z)^{-1} = 1 + \sum b_g z^{2g}$. Then we define:

$$\chi(y,z) = \frac{y + 1 - y^2 \frac{\partial \mathcal{H}}{\partial y}}{y\mathcal{S}(z)^2} \cdot \exp\left(y \left(\frac{z\mathcal{S}'(z)}{\mathcal{S}(z)} - \ln(\mathcal{S}(z)) - \mathcal{H} \right) \right).$$

Theorem 1.5. *For all $g \geq 1$, the orbifold Euler characteristic of the minimal stratum $\mathcal{M}_g(2g-1)$ is given by*

$$\chi(\mathcal{M}_g(2g-1)) = (2g-1)^{2g-1}[z^{2g}]\chi(2g-1, z).$$

This result simplifies drastically the computation of the Euler characteristic of the minimal strata from the formula shown in [CMZ20a] involving a sum over a high number of combinatorial graphs. In particular, this makes it feasible to explicitly compute the value of $\chi(\mathcal{M}_g(2g-1))$ for large values of g.

Finally, we have a spin analogue of Theorem 1.5. We define

$$\chi(\mathcal{M}_g(2g-1))^{\mathrm{spin}} := \chi(\mathcal{M}_g(2g-1)^{\mathrm{even}}) - \chi(\mathcal{M}_g(2g-1)^{\mathrm{odd}})$$

to be difference of the orbifold Euler characteristics of the even and odd components of the minimal stratum $\mathcal{M}_g(2g-1)$.

Theorem 1.6. *If Assumption 1.3 holds, then for all $g \geq 1$, we have*

$$\chi(\mathcal{M}_g(2g-1))^{\mathrm{spin}} = 2^{-g}(2g-1)^{(2g-1)}[z^{2g}]\chi^{\mathrm{spin}}(2g-1, z),$$

where we define:

$$\begin{aligned} \mathcal{H}^{\mathrm{spin}}(y,z) &= -\sum_{g\geq 1} (2g-1)! \frac{2^{2g-1}}{2^{2g-1}-1} b_g \left(\frac{z\mathcal{S}(z)}{y}\right)^{2g} \frac{\mathcal{S}((2g-1)z)}{\mathcal{S}(z)}, \\ \chi^{\mathrm{spin}}(y,z) &= \frac{y + 1 - y^2 \frac{\partial \mathcal{H}^{\mathrm{spin}}}{\partial y}}{y\mathcal{S}(z)^2 \cosh(z/2)^{-1}} \cdot \exp\left(y \left(\frac{z\mathcal{S}'(z)}{\mathcal{S}(z)} - \ln(\mathcal{S}(z)) - \mathcal{H}^{\mathrm{spin}} \right) \right). \end{aligned}$$

In Table 1 we present the orbifold Euler characteristic of the even and odd spin components of minimal strata of abelian differentials in low genera.

g	1	2	3	4	5	6
$\chi(\mathcal{M}_g(2g-1)^{\mathrm{even}})$	0	0	$-\frac{1}{84}$	$-\frac{269}{720}$	$-\frac{693}{40}$	$-\frac{76466}{63}$
$\chi(\mathcal{M}_g(2g-1)^{\mathrm{odd}})$	$-\frac{1}{12}$	$-\frac{1}{40}$	$-\frac{7}{72}$	$-\frac{5}{4}$	$-\frac{3933}{110}$	$-\frac{5841833}{3120}$

Table 1. Euler characteristics of even and odd components of some minimal holomorphic strata.

1.5. **Outlook and open questions.** A first natural question to ask is how Theorem 1.1 generalizes to arbitrary monomials in ψ-classes.

Question 1.7. *Given $g, n \geq 0$ and nonnegative integers $e_1, \ldots, e_n$ summing to $2g-3+n$, compute the function*

$$\mathbb{Z}^n \to \mathbb{Q}, \quad a \mapsto \int_{\overline{\mathcal{M}}_{g,n}} \mathrm{DR}_g(a) \cdot \psi_1^{e_1} \cdots \psi_n^{e_n} .$$

Restricted to vectors a with $|a| = 0$, this question was answered in [BSSZ15, Theorem 2]. The proof of this Theorem was achieved by using the semi-infinite wedge formalism and by computing integrals of ψ-classes on classical DR cycles as vacuum expectations (see [BSSZ15, Theorem 7]).

While the intersection numbers above are readily computable for many pairs (g, n) using the software `admcycles` [DSv20], it proved difficult to guess a formula generalizing the one presented in [BSSZ15]. Alternatively one could try to understand how to naturally account for the twisting parameter k in the semi-infinite wedge formalism. To our knowledge, such approach does not exist in the literature yet. A first step would be to define a "natural" operator acting on the infinite wedge space, whose vacuum expectation is given by the right-hand side of (4).

A second direction of study is the asymptotic behaviour of the orbifold Euler characteristic of minimal strata of differentials. From numerical experiments based on the formula given in Theorem 1.5, we propose the following conjecture.

Conjecture 1.8. *For all $g \geq 1$, the orbifold Euler characteristic $\chi(\mathcal{M}_g(2g-1))$ is negative. Moreover, there exist positive constants A and B such that*

$$A \frac{(2g-1)!}{(2g-1)^4} \leq -\chi(\mathcal{M}_g(2g-1)) \leq B \frac{(2g-1)!}{(2g-1)^3}$$

for all g.

Note that this asymptotic growth rate is much higher than the one of $\mathcal{M}_{g,1}$, which was shown in [HZ86] to be $(-1)^g \frac{(2g-1)!}{2(2\pi)^g}$. The conjecture is also a strong indication that, as in the case of $\mathcal{M}_{g,n}$, the cohomology of minimal strata is not spanned by tautological classes. However, in order to formally prove this, one would have to compute the topological Euler characteristic, and not the orbifold one. Note that in the case of $\mathcal{M}_{g,n}$, it was shown in [HZ86] that the growth rates of the topological and orbifold Euler characteristics agree.

The global geometry of strata of abelian differentials is not well understood. The above conjecture fits into the setting of two natural questions that are still open:

Question 1.9. *Is $\mathcal{M}_g(a)$ kähler-hyperbolic?*

The analogous result in the case of $\mathcal{M}_{g,n}$ was proven in [McM00]. This property would imply that the sign of the Euler characteristic of $\mathcal{M}_g(a)$ is $(-1)^{\dim(\mathcal{M}_g(a))}$. In particular, since the minimal strata $\mathcal{M}_g(2g-1)$ are always of odd dimension $2g-1$, this prediction is consistent with the first part of Conjecture 1.8.

Question 1.10. *Are the connected components of $\mathcal{M}_g(a)$ classifying spaces for their fundamental groups?*

This question was posed as a conjecture by Kontsevich and Zorich. The image of the fundamental group of the connected components of $\mathcal{M}_g(a)$ in the mapping class group was described in [CS20] to be a framed mapping class group. In the same work, it was shown to be a finitely generated infinite index subgroup of the associated mapping class group. However, the kernel of the natural map to the mapping class group is still ill-understood. Some analogous results appeared also in [Ham20]. One approach for *disproving* the conjecture of Kontsevich and Zorich, would be to use Conjecture 1.8 to show that the Euler characteristic of the minimal strata and of their the fundamental groups are different for large values of g. This is not possible until more information on the fundamental group of the strata is known.

From numerical experiments, using the formula given in Theorem 1.6, we also conjecture that the Euler characteristics of the odd and even components are asymptotically equivalent for large values of g. This is implied by the following stronger conjecture.

Conjecture 1.11. *There exists positive constants A and B such that*

$$A\frac{g}{2^g} < \frac{\chi(\mathcal{M}_g(2g-1))^{\mathrm{spin}}}{-\chi(\mathcal{M}_g(2g-1))} < B\frac{g^2}{2^g}$$

for all g. In particular $\chi(\mathcal{M}_g(2g-1))^{\mathrm{spin}} > 0$ and

$$\frac{\chi(\mathcal{M}_g(2g-1)^{\mathrm{even}})}{\chi(\mathcal{M}_g(2g-1)^{\mathrm{odd}})} \longrightarrow 1 \quad \textit{for} \;\; g \longrightarrow \infty.$$

1.6. **Organization of the paper.** We begin the paper by recalling in Section 2 the definition of double ramification cycles and their relation to the strata of k-differentials as well as the smooth compactification of these strata by the spaces of multi-scale differentials. In particular, in Section 2.3 we present our proposal for the spin double ramification cycles and the corresponding generalization of Conjecture A. In Section 3, we prove a *splitting formula* for ψ-classes on double ramification cycles, i.e. a family of relations between ψ-classes on double ramification cycles. The splitting formula is then used to show a list of properties of the functions $\mathcal{A}_g$ in Section 4 which uniquely determine these functions. We finish the proof of Theorem 1.1 by verifying these properties for the right-hand side of equation (3). We conclude the paper by discussing the spin refinements of our results in Section 5, and the application to the Euler characteristic of minimal strata of differentials in Section 6.

In Appendix A we collect some results about polynomiality properties of some formulas in our paper, which are used in the proofs of Section 4.

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2. Double ramification cycles and moduli spaces of multi-scale differentials

In this section we recall the definitions of the cycles $\mathrm{DR}_g(a)$ and moduli spaces of multi-scale differentials, which will be central in the rest of the paper. In particular we recall the notion of twisted and level graphs and we fix the notation that will be used in the next sections.

2.1. **Twisted and level graphs.** Let $g, n \geq 0$ with $2g-2+n>0$. A *stable graph* is the datum of

$$\Gamma = (V, H, g : V \to \mathbb{Z}_{\geq 0}, i : H \to H, \phi : H \to V, H^i \simeq [\![1, n]\!]),$$

where:

- The function i is an involution of H.
- The cycles of length 2 for i are called *edges* while the fixed points are called *legs*. We fix the identification of the set of legs with the set $[\![1, n]\!]$ of integers from 1 to n.
- An element of V is called a *vertex* and for a half-edge h with $\phi(h) = v$ we say that h is *incident* to v. We denote by $H_\Gamma(v)$ the set of half-edges incident to v and by $n(v)$ the *valency* of the vertex v, i.e. the cardinality of $H_\Gamma(v)$.
- For all vertices v we have $2g(v) - 2 + n(v) > 0$.
- The genus of the graph, defined as
$$g(\Gamma) = h^1(\Gamma) + \sum_{v \in V} g(v), \text{ with } h^1(\Gamma) = |E| - |V| + 1$$
is equal to g.
- The graph is connected.

A stable graph determines a moduli space $\overline{\mathcal{M}}_\Gamma = \prod_{v\in V} \overline{\mathcal{M}}_{g(v),n(v)}$ and a morphism $\zeta_\Gamma : \overline{\mathcal{M}}_\Gamma \to \overline{\mathcal{M}}_{g,n}$ (see e.g. [GP03, Appendix A]).

We fix a value of $k \geq 0$ and a vector a such that $|a| = k(2g-2+n)$.

Definition 2.1 ([FP18])**.** A *twist* (compatible with a) on a stable graph Γ is a function $I : H \to \mathbb{R}$ satisfying:

- For all $v \in V$, we have
$$\sum_{h \in \phi^{-1}(v)} I(h) = k(2g(v) - 2 + n(v)).$$
- If (h, h') is an edge of Γ, then we have $I(h) = -I(h')$.
- If (h_1, h_1') and (h_2, h_2') are edges between the same vertices v, v', then $I(h_1) \geq 0 \Leftrightarrow I(h_2) \geq 0$. In which case we denote $v \geq v'$.
- The relation $\geq$ defines a partial order on the set of vertices.
- The twist at the leg with label i has value a_i.

Given a twisted graph (Γ, I), we define its multiplicity as

$$m(\Gamma, I) = \prod_{(h,h') \in E(\Gamma)} \sqrt{-I(h)I(h')}\,.$$

We now introduce the objects that parametrize the boundary component of the moduli space of multi-scale differentials.

Let (Γ, I) be a twisted graph. A *level function* of depth $L > 0$ on (Γ, I) is a surjective function $l : V(\Gamma) \to \{0, -1, \dots, -L\}$ satisfying $l(v_1) \geq l(v_2)$ if $v_1 \geq v_2$, for any v_1 and v_2 in $V(\Gamma)$,

Definition 2.2. A *level graph* $\overline{\Gamma} = (\Gamma, I, l)$ is the datum of a twisted graph together with a level function. We denote by $\mathrm{LG}(g, a)$ the set of level graphs for the vector a and with no horizontal edges (edge between vertices of the same level) and $\mathrm{LG}_L(g, a)$ the set of such level graphs of depth L.

If $\overline{\Gamma}$ is a level graph in $\mathrm{LG}_L(g, a)$, and $1 \leq i \leq L$, then we denote by $m(\overline{\Gamma})^{[i]}$ and $\ell(\overline{\Gamma})^{[i]}$ the product and lcm of the twists at all the edges crossing the level-passage between levels $-i+1$ and i. Here, the twist of an edge $e = \{h, h'\}$ is the positive integer $|I(h)| = |I(h')|$. Finally we define $m(\overline{\Gamma}) = \prod_i m(\overline{\Gamma})^{[i]}$, and $\ell(\overline{\Gamma}) = \prod_i \ell(\overline{\Gamma})^{[i]}$.

2.2. **A formula for twisted double ramification cycles.** The double ramification cycle $\mathrm{DR}_g(a)$ on $\overline{\mathcal{M}}_{g,n}$ is a cycle compactifying the condition $\omega_{\log}^{\otimes k} \cong \mathcal{O}_C(a_1 x_1 + \ldots + a_n x_n)$ on $\mathcal{M}_{g,n}$. Many geometric approaches have been proposed to make this statement precise (see [HS21, Section 1.6] for an overview) and over the last years, all of them have been shown to be equivalent.

A particularly explicit approach is to express the cycle $\mathrm{DR}_g(a)$ in terms of the generators of the tautological ring of $\overline{\mathcal{M}}_{g,n}$. To state the corresponding formula below, we write $\mathcal{G}_{g,n}$ for the set of stable graphs Γ of genus g with n legs. Fix a vector $a \in \mathbb{Z}^n$ such that the number $k = |a|/(2g-2+n)$ is an integer. Then for a stable graph Γ, an *admissible weighting* modulo r (with respect to a) is a map $w : H(\Gamma) \to \{0, \dots, r-1\}$ satisfying:

a) For every vertex $v \in V(\Gamma)$, we have
$$\sum_{h \in H_\Gamma(v)} w(h) \equiv k(2g-2+n) \bmod r\,.$$

b) For every edge $e = (h, h') \in E(\Gamma)$, we have
$$w(h) + w(h') \equiv 0 \bmod r\,.$$

c) For every $i = 1, \dots, n$ and h_i the half-edge associated to the i-th marking, we have
$$w(h_i) \equiv a_i \bmod r\,.$$

We write $W_{\Gamma,r}(a)$ for the set of admissible weightings modulo r on Γ. Then, we define a mixed-degree tautological class $P_g^{r,\bullet}(a)$ on $\overline{\mathcal{M}}_{g,n}$ by the formula

$$P_g^{r,\bullet}(a) = r^{2g} \cdot \sum_{\Gamma \in \mathcal{G}_{g,n}} \sum_{w \in W_{\Gamma,r}(a)} \frac{r^{-h^1(\Gamma)}}{|\mathrm{Aut}(\Gamma)|} \xi_{\Gamma *} \mathrm{Cont}_{a,\Gamma,w,r}\,, \tag{6}$$

where the class $\mathrm{Cont}_{a,\Gamma,w,r}$ on the domain $\overline{\mathcal{M}}_\Gamma$ of the gluing map ξ_Γ is defined by

$$\begin{aligned}\mathrm{Cont}_{a,\Gamma,w,r} = &\prod_{v\in V(\Gamma)} \exp(-\sum_{m\geq 1}(-1)^{m-1}\frac{B_{m+1}(k/r)}{m(m+1)}\kappa_m(v))\\ &\cdot \prod_{i=1,\dots,n} \exp(\sum_{m\geq 1}(-1)^{m-1}\frac{B_{m+1}(a_i/r)}{m(m+1)}\psi_i^m)\\ &\cdot \prod_{\substack{e\in E(\Gamma)\\ e=(h,h')}} \frac{1-\exp(\sum_{m\geq 1}(-1)^{m-1}\frac{B_{m+1}(w(h)/r)}{m(m+1)}(\psi_h^m-(-\psi_{h'})^m)}{\psi_h+\psi_{h'}}.\end{aligned}$$

For sufficiently large values of r, this class is a mixed degree tautological class on $\overline{\mathcal{M}}_{g,n}$ whose coefficients are polynomials in r (see [JPPZ17]). We denote by $P_g^\bullet(a)$ the evaluation of this class by substituting $r=0$ in its coefficients. Then we define the double ramification cycle as the multiple

$$\mathrm{DR}_g(a) = 2^{-g}P_g^g(a)$$

of the Chow degree g part of this cycle.

If $k>0$ and $a \notin k\mathbb{Z}_{>0}^n$, the double ramification cycle $\mathrm{DR}_g(a)$ has a geometric interpretation as the weighted fundamental class of the locus of twisted k-differentials in $\overline{\mathcal{M}}_{g,n}$. This relationship was first conjectured in [FP18] for $k=1$ and in [Sch18] for $k>1$ and was proved in [HS21, BHP+20]. We recall this expression here.

Definition 2.3. A *simple star graph* (Γ, I) is a twisted graph such that

- the vertices of Γ consist of a unique *central vertex* v_0 and a set V_{Out} of outlying vertices, such that all edges of Γ go from v_0 to one of the outlying vertices,
- the twists at half-edges adjacent to outlying vertices are positive and divisible by k.

Two simple star graphs are called isomorphic if there is an isomorphism of the underlying stable graphs sending the central vertex to the central vertex and respecting the respective twists. Denote by $\mathrm{Star}_g(a)$ the set of isomorphism classes of simple star graphs for the given genus g and vector a.

Given (Γ, I) and $v\in V(\Gamma)$ a vertex, we denote by $I(v)$ the vector of integers $I(h)$ indexed by the half-edges h at v. Then we define

$$\overline{\mathcal{M}}_{\Gamma,I} = \overline{\mathcal{M}}_{g(v_0)}(I(v_0)) \times \prod_{v\in V_{\mathrm{Out}}} \overline{\mathcal{M}}_{g(v)}(I(v)/k)\,. \tag{7}$$

The space $\overline{\mathcal{M}}_{\Gamma,I}$ is naturally a closed substack of $\overline{\mathcal{M}}_\Gamma$.

With this notation in place, we are ready to state the relationship between the strata of k-differentials and the double ramification cycle.

Theorem 2.4 ([HS21, BHP+20])**.** *Let $g,n\geq 0$, $k>0$, and $a\in\mathbb{Z}^n$ such that $|a| = k(2g-2+n)$ and such that $a\notin k\mathbb{Z}_{>0}^n$. Then we have*

$$\mathrm{DR}_g(a) = \sum_{(\Gamma,I)\in\mathrm{Star}_g(a)} \frac{m(\Gamma,I)}{k^{|V_{\mathrm{Out}}|}|\mathrm{Aut}(\Gamma,I)|}\cdot \zeta_{\Gamma *}[\overline{\mathcal{M}}_{\Gamma,I}]. \tag{8}$$

2.3. **The spin double ramification cycle.** In this section, we give an explicit proposal for the cycle $\mathrm{DR}_g^{\mathrm{spin}}$ from Assumption 1.3 as well as a conjecture generalizing Conjecture A of [FP18] and [Sch18] to the spin setting. The proposal for $\mathrm{DR}_g^{\mathrm{spin}}$ is inspired by recent work [GKL21] on spin Hurwitz numbers. Its formula consists in a small modification of the original formulas for the double ramification cycle presented above.

First we define the spin analogue of Pixton's class above. Let r be an even number, and $a \in \mathbb{Z}^n$ a vector of odd integers such that the number $k = |a|/(2g-2+n)$ is an odd integer. Given a stable graph Γ, we write $W_{\Gamma,r}(a)^{\mathrm{odd}}$ for the set of admissible weightings modulo r on Γ satisfying that all numbers $w(h)$, $h \in H(\Gamma)$, are odd. It is easy to see that

- for Γ a tree there exists a unique admissible weighting modulo r and it is automatically odd,
- for arbitrary Γ, the cardinality of $W_{\Gamma,r}(a)^{\mathrm{odd}}$ is precisely $(r/2)^{h^1(\Gamma)}$.

Then, we define a mixed-degree tautological class $P_g^{r,\mathrm{spin},\bullet}(a)$ on $\overline{\mathcal{M}}_{g,n}$ by the formula

$$P_g^{r,\mathrm{spin},\bullet}(a) = r^{2g}2^{-g} \cdot \sum_{\Gamma \in \mathcal{G}_{g,n}} \sum_{w \in W_{\Gamma,r}(a)^{\mathrm{odd}}} \frac{(r/2)^{-h^1(\Gamma)}}{|\mathrm{Aut}(\Gamma)|} \xi_{\Gamma *}\mathrm{Cont}_{a,\Gamma,w,r}\,, \tag{9}$$

where the class $\mathrm{Cont}_{a,\Gamma,w,r}$ is the class defined in the previous section. Using similar arguments as presented in [JPPZ17, Appendix A], one sees that the cycle $P_g^{r,\mathrm{spin},\bullet}(a)$ become polynomial in r for sufficiently large (even) r. Denote by $P_g^{g,\mathrm{spin}}(a)$ the value of the degree g part of this polynomial at $r=0$. Then we propose that the spin double ramification cycle should be given by

$$\mathrm{DR}_g^{\mathrm{spin}}(a) = 2^{-g} P_g^{g,\mathrm{spin}}(a)\,. \tag{10}$$

As mentioned before, the formula above is inspired by the paper [GKL21]. There, the authors introduce a spin Chiodo class $C^{\vartheta}(r,k;\widetilde{a})$, where the modified vector $\widetilde{a}$ is defined by the convention $2\widetilde{a}_i + 1 = a_i$. The comparison to the class $P_g^{r,\mathrm{spin},\bullet}(a)$ is

$$C^{\vartheta}(r,k;\widetilde{a}) = r^{-1} P_g^{r,\mathrm{spin},\bullet}(a)\,,$$

see [GKL21, Proposition 9.7, Proposition 9.20]. We used an implementation by Danilo Lewański of the formula for $\mathrm{DR}_g^{\mathrm{spin}}(a)$ in the software `admcycles` [DSv20] to verify that for the cases

$$(g=1, n \leq 4), (g=2, n \leq 2), (g=3, n=1)$$

the top-ψ intersection number of the corresponding cycle agrees with the prediction from Theorem 1.4 for *all* admissible input vectors a.

Concerning the statement in Assumption 1.3 that the cycle (10) is a polynomial in a of degree $2g$, it will be proved in the paper [PZ] by Pixton and Zagier, alongside the corresponding statement for the classical double ramification cycle. Furthermore, the statement $\pi^*\mathrm{DR}_g(a)^{\mathrm{spin}} = \mathrm{DR}_g(a,k)^{\mathrm{spin}}$ can be shown from the formula above using a short direct computation.

As for the remaining parts of Assumption 1.3, namely the relations between top ψ_1-intersection numbers of $\mathrm{DR}_g^{\mathrm{spin}}(a)$ and $[\overline{\mathcal{M}}_g(a)]^{\mathrm{spin}}$ (for suitable k, a), these would be implied by a spin variant of Conjecture A, which we present in the following. For this, we denote by $\mathrm{Star}_g(a)^{\mathrm{odd}}$ the set of star graphs with odd values of twists at all half-edges. Given such a pair $(\Gamma, I) \in \mathrm{Star}_g(a)^{\mathrm{odd}}$ we write

$$[\overline{\mathcal{M}}_{\Gamma,I}]^{\mathrm{spin}} = [\overline{\mathcal{M}}_{g(v_0)}(I(v_0))]^{\mathrm{spin}} \otimes \prod_{v \in V_{\mathrm{Out}}} [\overline{\mathcal{M}}_{g(v)}(I(v)/k)]^{\mathrm{spin}} . \tag{11}$$

Then we conjecture the following relationship between the spin cycles of the strata of k-differentials and the spin double ramification cycle above.

Conjecture 2.5. *Let $g, n \geq 0$, $k > 0$, and $a \in \mathbb{Z}^n$ such that $|a| = k(2g-2+n)$ satisfying that $a \notin k\mathbb{Z}^n_{>0}$ and that all entries of a and k are odd. Then we have*

$$\sum_{(\Gamma,I) \in \mathrm{Star}_g(a)^{\mathrm{odd}}} \frac{m(\Gamma, I)}{k^{|V_{\mathrm{Out}}|}|\mathrm{Aut}(\Gamma, I)|} \cdot \zeta_{\Gamma*}[\overline{\mathcal{M}}_{\Gamma,I}]^{\mathrm{spin}} = \mathrm{DR}_g^{\mathrm{spin}}(a) . \tag{12}$$

Proposition 2.6. *Assuming Conjecture 2.5 is true, the class $\mathrm{DR}_g^{\mathrm{spin}}(a)$ satisfies properties (3) and (4) of Assumption 1.3.*

Proof. To show property (3) we intersect both sides of the equality (12) with the monomial P in classes ψ_i for which $a_i \notin k\mathbb{Z}_{>0}$. This last assumption on the a_i forces all markings associated to classes ψ_i appearing in P to be on the central vertex of any graph $(\Gamma, I) \in \mathrm{Star}_g(a)^{\mathrm{odd}}$. If Γ is non-trivial, a short computation shows that the class $[\overline{\mathcal{M}}_{g(v_0)}(I(v_0))]^{\mathrm{spin}}$ inserted at the central vertex has dimension strictly less than the degree $2g-3+n$ of P. Thus for the intersection number of P with the left-hand side of (12), the only surviving term comes from the trivial star graph. This gives the left-hand side of the equation from property (3) and thus finishes the proof. Finally, property (4) can be shown by similar arguments as appear in [Sau20, Section 2.2]. □

Apart from theoretical evidence for Conjecture 2.5 which we discuss below, there are a few cases where it can be verified directly.

In genus $g = 0$ it is trivial: a spin bundle $\mathcal{L}$ on $\mathbb{P}^1$ is always even and so both sides of the conjecture equal the fundamental class of $\overline{\mathcal{M}}_{0,n}$.

In genus $g = 1$, for a point $(C, x_1, \ldots, x_n) \in \mathcal{M}_1(a)$ with $k = 1$ and a odd, the spin bundle

$$L = \mathcal{O}_C\left(\frac{a_1 - k}{2}x_1 + \ldots + \frac{a_n - k}{2}x_n\right)$$

on an elliptic curve C is a 2-torsion line bundle. We can distinguish two cases: for $L = \mathcal{O}_C$ it has precisely one section and thus it is odd, whereas for $L \neq \mathcal{O}_C$ it has no section and thus is even. Therefore the odd components of $\mathcal{M}_1(a)$ are precisely the loci where L is trivial and thus they are given by $\mathcal{M}_1((a_1-k)/2, \ldots, (a_n-k)/2)$. Using that $[\overline{\mathcal{M}}_g(a)]^{\mathrm{spin}} = [\overline{\mathcal{M}}_g(a)] - 2 \cdot [\overline{\mathcal{M}}_g(a)]^{\mathrm{odd}}$ we then have

$$[\overline{\mathcal{M}}_1(a_1, \ldots, a_n)]^{\mathrm{spin}} = [\overline{\mathcal{M}}_1(a_1, \ldots, a_n)] - 2 \cdot [\overline{\mathcal{M}}_1(\frac{a_1-k}{2}, \ldots, \frac{a_n-k}{2})] . \tag{13}$$

The cycles $[\overline{\mathcal{M}}_1(a)]$ themselves are determined by the original Conjecture A. Thus in genus $g = 1$ one can explicitly compute both sides of the equality (12) since in the sum over star graphs, we only need to compute the cycles $[\overline{\mathcal{M}}_0(a)]^{\mathrm{spin}} = [\overline{\mathcal{M}}_0(a)]$ and $[\overline{\mathcal{M}}_1(a)]^{\mathrm{spin}}$. A short calculation (similar to one appearing in [Sch18, Section 3.3.1]) then verifies Conjecture 2.5.

Finally, using the software `admcycles` [DSv20], the conjecture can be checked for $g = 2$, $a = (5, -1)$. In the sum over star graphs (Γ, I), the only genus 2 spin cycles that appear as factors of $[\overline{\mathcal{M}}_{\Gamma,I}]^{\mathrm{spin}}$ are $[\overline{\mathcal{M}}_2(3)]^{\mathrm{spin}}$ and $[\overline{\mathcal{M}}_2(5,-1)]^{\mathrm{spin}}$.

In these cases, it follows from the classification of the connected components of $\overline{\mathcal{M}}_2(3)$ and $\overline{\mathcal{M}}_2(5,-1)$ presented in [KZ03, Boi15] that the spaces have precisely two components: a hyperelliptic one of odd parity and a non-hyperelliptic one of even parity. This implies that

$$[\overline{\mathcal{M}}_2(3)]^{\mathrm{spin}} = [\overline{\mathcal{M}}_2(3)] - 2\cdot[\overline{\mathrm{Hyp}}_{2,1}], \quad [\overline{\mathcal{M}}_2(5,-1)]^{\mathrm{spin}} = [\overline{\mathcal{M}}_2(5,-1)] - 2\cdot[\overline{\mathrm{Hyp}}_{2,2}]\,,$$

where $\overline{\mathrm{Hyp}}_{2,1} \subseteq \overline{\mathcal{M}}_{2,1}$ and $\overline{\mathrm{Hyp}}_{2,2} \subseteq \overline{\mathcal{M}}_{2,2}$ are the loci of (hyperelliptic) curves with one or two marked Weierstrass points. Their fundamental classes can be computed using the methods described in [SvZ20]. With these inputs, all terms appearing in Conjecture 2.5 can be computed as tautological classes and it is verified that the claimed equality follows from known tautological relations.

A natural path towards a proof of Conjecture 2.5 should be as follows:

(i) Consider a compactification $\overline{\mathcal{S}}_{g,n}$ of the moduli space

$$\mathcal{S}_{g,n} = \{(C, x_1, \ldots, x_n, \mathcal{L}) : \mathcal{L}^{\otimes 2} \cong \omega_C\}$$

of spin curves (see [Chi08, AJ03]). As seen before, this space decomposes into odd and even components, according to the parity of $\mathcal{L}$, and we define

$$[\overline{\mathcal{S}}_{g,n}]^{\mathrm{spin}} = [\overline{\mathcal{S}}_{g,n}]^{\mathrm{even}} - [\overline{\mathcal{S}}_{g,n}]^{\mathrm{odd}}\,.$$

(ii) On the other hand, we can consider the universal Picard stack $\mathcal{P}ic_{g,n}$ over $\overline{\mathcal{S}}_{g,n}$ parameterizing a curve C in $\overline{\mathcal{S}}_{g,n}$ together with a line bundle $\mathcal{M}$ on C. Let $e \subseteq \mathcal{P}ic_{g,n}$ be the codimension g locus where $\mathcal{M} = \mathcal{O}_C$ is trivial, and denote by $\overline{e}$ its closure.

Given k odd and a vector a of odd integers summing to $k(2g-2+n)$, we can consider the section $\sigma_a : \overline{\mathcal{S}}_{g,n} \to \mathcal{P}ic_{g,n}$ given by

$$\mathcal{M} = \mathcal{L}^\vee \otimes \omega_C^{\otimes(-k+1)/2} \otimes \mathcal{O}_C\left(\sum_{i=1}^n \frac{a_i - k}{2} x_i\right).$$

We define the cycle

$$\widehat{\mathrm{DR}}_g(a) = \sigma^*[\overline{e}] \in A^g(\overline{\mathcal{S}}_{g,n})\,.$$

Intuitively, this cycle compactifies the condition

$$\mathcal{L} \cong \omega_C^{\otimes(-k+1)/2} \otimes \mathcal{O}_C\left(\sum_{i=1}^n \frac{a_i - k}{2} x_i\right).$$

that we used in the definition of the spin refinement of the strata of k-differentials. We then expect that the machinery of the paper [HS21] can be adapted to show that the cycle

$$F_*\left(\widehat{\mathrm{DR}}_g(a) \cdot [\overline{\mathcal{S}}_{g,n}]^{\mathrm{spin}}\right)$$

is precisely given by the linear combination of cycles on the left-hand side of (12), with the numbers $m(\Gamma, I)$ being related to intersection multiplicities of the section σ with the locus $\overline{e}$.

(iii) On the other hand, generalizing the machinery of [BHP$^+$20], the class $[\overline{e}] \in A^g(\mathcal{P}ic_{g,n})$ should have a formula in the tautological ring of $\mathcal{P}ic_{g,n}$ which is structurally very similar to Pixton's formula for the classical double ramification cycle. Pulling this back under σ we obtain a formula for

$\widehat{\mathrm{DR}}_g(a)$ and a computation analogous to [GKL21, Proposition 9.18] should imply

$$F_*\left(\widehat{\mathrm{DR}}_g(a)\cdot[\overline{\mathcal{S}}^{\mathrm{spin}}_{g,n}]\right)=\mathrm{DR}_g(a)^{\mathrm{spin}},$$

which would conclude the proof.

We plan to pursue these directions further in the future.

2.4. **The moduli space of multi-scale differentials.** We recall now the definition and some features of the moduli space of multi-scale differentials, a smooth modular compactification of strata of differentials with normal crossing boundary divisors.

2.4.1. *The incidence variety compactification.* Given a vector $P=(p_1,\dots,p_n)$ of positive integers, we denote by $\overline{\Omega}^k_{g,n}(P)$ the vector bundle on $\overline{\mathcal{M}}_{g,n}$ defined as

$$\pi_*\omega_{\log}^{\otimes k}(p_1\sigma_1+\dots,p_n\sigma_n)$$

where we recall that $\pi:\overline{\mathcal{C}}_{g,n}\to\overline{\mathcal{M}}_{g,n}$ is the universal curve and the $\sigma_i:\overline{\mathcal{M}}_{g,n}\to\overline{\mathcal{C}}_{g,n}$ are the sections associated to the markings. If a is a vector of integers satisfying $|a|=k(2g-2+n)$, then for a sufficiently large value of P, we define $\Omega_g(a)$ as the sub-space of $\overline{\Omega}^k_{g,n}(P)$ of smooth curves with a k-differentials with singularities prescribed by a. We denote by $\mathbb{P}\overline{\Omega}_g(a)$ the Zariski closure of $\mathbb{P}\Omega_g(a)$ in $\mathbb{P}\overline{\Omega}^k_{g,n}(P)$ (the geometry of this space does not depend on the choice of P). This space is the incidence variety compactification of $\mathcal{M}_g(a)$ (which is not smooth). See [BCG$^+$19a] for the description of the boundary of this compactification.

2.4.2. *The moduli space of multi-scale differentials.* The incidence variety admits a modular desingularization $\mathbb{P}\Xi_g(a)\to\mathbb{P}\overline{\Omega}_g(a)$, the *moduli space of multi-scale differentials*, which is defined in [BCG$^+$19b] in the case $k=1$ and in [CMZ19] for higher values of k. We denote by $p:\mathbb{P}\Xi_g(a)\to\overline{\mathcal{M}}_g(a)$ the morphism defined as the composition of the desingularization and the forgetful morphism $\mathbb{P}\overline{\Omega}_g(a)\to\overline{\mathcal{M}}_g(a)$. This morphism restricts to an isomorphism over $\mathcal{M}_g(a)$.

The boundary $\mathbb{P}\Xi_g(a)\backslash\mathcal{M}_g(a)$ is a simple normal crossing divisor. The boundary components of this space $\mathbb{P}\Xi_g(a)$ are parametrized by level graphs, as introduced in Definition 2.2. The sum of the depth and the number of horizontal edges of a level graph is the codimension of the associated boundary component. In particular divisors are indexed by either level graphs of depth 0 with 1 (horizontal) edge, or level graphs in $\mathrm{LG}_1(g,a)$. We will denote by $\mathbb{P}\Xi(\overline{\Gamma})\subseteq\mathbb{P}\Xi_g(a)$ the boundary component corresponding to the level graph $\overline{\Gamma}$.

If $\overline{\Gamma}$ is a graph in $\mathrm{LG}_L(g,a)$, then the associated boundary component $\mathbb{P}\Xi(\overline{\Gamma})$ parametrizes *multi-scale differentials* compatible with $\overline{\Gamma}$ modulo an equivalence relation given by the action of a torus, called *level rotation torus*. We explain now the details that we will need for the case of $k=1$ in Section 5.1, where we investigate the spin parity of some boundary divisors. We refer to [BCG$^+$19b] for the full description of this compactification.

An *abelian twisted differential* is a tuple $(\omega_v)_{v\in V(\overline{\Gamma})}$ of abelian differentials on each component of the stable curve defined by $\overline{\Gamma}$. We say that an abelian twisted differential is compatible with the level graph $\overline{\Gamma}$ if each ω_v has vanishing orders prescribed by the twists of $\overline{\Gamma}$ and if it satisfies the *global residue condition* (GRC) defined in [BCG$^+$19a, Definition 1.4]. An abelian multi-scale differential compatible with $\overline{\Gamma}$ is a twisted differential compatible with $\overline{\Gamma}$ together with the choice of a prong matching for each non-horizontal edge of Γ. A *prong matching* on a vertical edge of $\overline{\Gamma}$ is a bijection between the horizontal directions defined the twisted differentials on the components of the stable curve at the upper and lower ends of the edge.

If we denote by $L(\overline{\Gamma})$ the depth of $\overline{\Gamma}$, the equivalence relation is given by the action of $\mathbb{C}^{L(\overline{\Gamma})}$ which rescales the differentials on each level and simultaneously acts by fractional Dehn twists on the prong-matchings. The subgroup $\mathrm{Tw}_{\overline{\Gamma}} \subseteq \mathbb{C}^{L(\overline{\Gamma})}$ acting trivially on the prong matchings is called the Twist group, and so the action factors through the quotient $T_{\overline{\Gamma}} = \mathbb{C}^{L(\overline{\Gamma})}/\mathrm{Tw}_{\overline{\Gamma}}$, which is called the level rotation torus. Two multi-scale differentials compatible with $\overline{\Gamma}$ are defined to be equivalent, if they differ by the action of $T_{\overline{\Gamma}}$.

Now that we have explained which objects are parametrized by boundary divisors, we would like to recall that the boundary divisors $\mathbb{P}\Xi(\overline{\Gamma})$ are commensurable (see [CMZ20a, Prop. 4.4] for the details) to the product of lower dimensional spaces of multi-scale differentials, one for each level. We will denote by $\mathbb{P}\Xi(\overline{\Gamma})^{[i]}$ the moduli-space of multi-scale differentials defined by the i-th level of $\overline{\Gamma}$. More specifically, $\mathbb{P}\Xi(\overline{\Gamma})^{[i]}$ is the multi-scale differential compactification of the stratum

$$\mathbb{P}\Big(\prod_{v\in\ell^{-1}(i)} \Omega_{g(v)}(I(v))\Big)^{\mathfrak{R}_i}$$

where we denoted by $\mathfrak{R}_i$ the condition on residues induced by the GRC.

In the case of $k > 1$, the multi-scale compactification of a given stratum of k-differentials can be defined (up to stacky issues) as the closure of the embedding of the stratum into the corresponding stratum of abelian differentials obtained after the k-canonical covering construction. The main difference with respect to the abelian case is that for the same level graph, there can be different possibilities for the residue conditions $\mathfrak{R}_i$, so that each level stratum can have more irreducible components. Another source for new components of level strata in the case of $k > 1$ is given by the possibility of having to consider components given by strata of d-th power of k/d-differentials. All these new components are exactly indexed by k-canonical covers of the level graphs $\overline{\Gamma}$ (see [CMZ19]). However, even though the action of the level rotation torus is a priori more complicated to describe for $k > 1$, the most important remark to highlight is the fact that the torus still acts level-wise and there is a well-defined notion of prong-matching equivalence classes on $\overline{\Gamma}$, so we don't need to analyze the more refined classification by k-canonical covers.

Remark 2.7. We will not recall the GRC here but we mention the following facts that will be used below.

- If every connected component of the subgraph above level i contains at least one half-edge with a twist not in $k\mathbb{Z}_{>0}$, then the residue condition $\mathfrak{R}_i$ is empty.
- If $k = 1$, $n = 1$, and $\overline{\Gamma}$ is a 2-level graph of compact type, i.e. with only separating edges, then the GRC states that all residues at the poles of the multi-scale differential on level -1 are trivial.

2.4.3. *Intersection theory on the space of multi-scale differentials.* As $\mathbb{P}\overline{\Omega}_g(a)$ is subspace of a projective bundle, it has a tautologial bundle $\mathcal{O}(-1)$. We denote by

$\eta \in A^1(\mathbb{P}\Xi_g(a))$ the Chern class of the pull-back of this line bundle. We will also denote by $\psi_i \in A^1(\mathbb{P}\Xi_g(a))$ the pull-back of the ψ-classes from $\overline{\mathcal{M}}_g(a)$.

First of all we recall a relation between the tautological class η and the ψ-classes. In the case $k = 1$, the following statement was proven in [Sau19, Theorem 6(1)] for the incidence variety compactification and in [CMZ20a, Prop. 8.2] for the space of multi-scale differentials. The case of general $k \geq 1$ is treated in [Sau20, Theorem 3.12] in the context of the incidence variety compactification, but the proof directly shows the following statement about the space of multi-scale differentials.

Proposition 2.8. *For all $\overline{\Gamma} \in \mathrm{LG}_1(g,a)$, and all irreducible components D of $\mathbb{P}\Xi(\overline{\Gamma})$, there exists a rational number $m(D)$ satisfying the following conditions:*

- *for all $1 \leq i \leq n$, we have*

$$\eta = a_i\psi_i \ - \sum_{\overline{\Gamma} \in {}_i\mathrm{LG}_1(g,a)} \sum_D m(D) \cdot \zeta_{D*}\left([D^{[0]}] \otimes [D^{[-1]}]\right), \tag{14}$$

  where the set ${}_i\mathrm{LG}_1(g,a)$ consists of the two-level graphs $\overline{\Gamma}$ without horizontal edges where the i-th half-edge is on level -1, and $D^{[i]}$ are the i-th level strata induced by D.
- *if $k = 1$, or if all vertices of $\overline{\Gamma}$ have a half-edge of order not divisible by k, then $m(D) = m(\overline{\Gamma})/|\mathrm{Aut}(\overline{\Gamma})|$.*

Finally we recall the relation between the class of the subspace cut out by a residue condition and other tautological classes in the case of $k = 1$. As before, the following relation was proven in [Sau19, Prop. 7.6] for the incidence variety compactification and in [CMZ20a, Prop. 8.3] for the space of multi-scale differentials.

Proposition 2.9. *Let $k = 1$ and let $\mathbb{P}\Xi_g^{\mathfrak{R}}(a)$ be a stratum cut out by a set of residue conditions $\mathfrak{R}$. Assume that $\mathbb{P}\Xi_g^{\mathfrak{R}}(a)$ is a divisor in the stratum $\mathbb{P}\Xi_g^{\mathfrak{R}_0}(a)$ cut out by a smaller set $\mathfrak{R}_0$ of residue conditions obtained by removing one condition from $\mathfrak{R}$. Then we have the following relation*

$$[\mathbb{P}\Xi_g^{\mathfrak{R}}(a)] = \ -\eta \ - \sum_{\overline{\Gamma} \in \mathrm{LG}_{1,\mathfrak{R}}(g,a)} \ell(\overline{\Gamma}) \cdot [\mathbb{P}\Xi(\overline{\Gamma})] \tag{15}$$

in $A^1(\mathbb{P}\Xi_g^{\mathfrak{R}_0}(a))$, where $\mathrm{LG}_{1,\mathfrak{R}}(g,a)$ is the union of the sets of non-horizontal two-level graphs where the GRC on top level induced by $\mathfrak{R}$ does no longer introduce an extra condition and the set of non-horizontal two-level graphs where all the half-edges involved in the condition forming $\mathfrak{R} \setminus \mathfrak{R}_0$ go to lower level.

In order to convert the previous $\ell(\overline{\Gamma})$ coefficients when considering the commensurability diagram involving a level graph $\mathbb{P}\Xi(\overline{\Gamma})$ without horizontal edges and the product of all its level strata $\mathbb{P}\Xi(\overline{\Gamma})^{[i]}$, we recall that we have to multiply by

$$\frac{m(\overline{\Gamma})}{|\mathrm{Aut}(\overline{\Gamma})|\ell(\overline{\Gamma})}. \tag{16}$$

This was proven in [CMZ20a, Lemma 4.5].

3. Splitting formulas for ψ-classes on double ramification cycles

The purpose of this section is to prove the *splitting formula for ψ-classes on double ramification cycles*, i.e. a family of relations between ψ-classes on double ramification cycles (see Proposition 3.1 below). Theorem 2.4 will play a key-role in the proof as it allows to reduce intersection with double ramification cycles to intersection with strata of k-differentials.

To state this formula, we introduce the following notation: $\mathrm{LG}_1^2(g,a)$ is the set of level graphs with exactly 2 vertices v_0 of level 0 and v_{-1} of level -1, and no horizontal edges.

Proposition 3.1 (Splitting formula for ψ-classes on double ramification cycles)**.** *Let $g,n \geq 0$ with $2g-2+n>0$, let $a=(a_1,\dots,a_n)\in\mathbb{Z}^n$ with $\sum_{i=1}^n a_i = k(2g-2+n)$. Then for any two different elements $s,t\in\{1,\dots,n\}$, we have*

$$\begin{aligned}&(a_s\psi_s - a_t\psi_t)\mathrm{DR}_g(a_1,\dots,a_n)\\ &= \sum_{(\overline{\Gamma},I)\in \mathrm{LG}_1^2(g,a)} f_{s,t}(\overline{\Gamma})\frac{m(\overline{\Gamma})}{|\mathrm{Aut}(\overline{\Gamma})|}\cdot \zeta_{\Gamma *}\left(\mathrm{DR}_{g_0}(I(v_0))\otimes \mathrm{DR}_{g_{-1}}(I(v_{-1}))\right).\end{aligned} \tag{17}$$

Here, the function $f_{s,t}$ is defined by

$$f_{s,t}(\overline{\Gamma}) = \begin{cases} 0 & \text{if } s \text{ and } t \text{ belong to the same vertex,}\\ 1 & \text{if } s \text{ belong to } v_{-1} \text{ and } t \text{ to } v_0,\\ -1 & \text{otherwise.}\end{cases}$$

3.1. **Splitting formulas for strata of k-differentials.** In order to prove Proposition 3.1, we will show that a similar statement holds when we replace $\mathrm{DR}_g(a)$ by $[\overline{\mathcal{M}}_g(a)]$. An analogous splitting formula was proved in the case $k=1$ in [Sau19].

Lemma 3.2. *Let $g,n\geq 0$, $k>0$, and $a\in\mathbb{Z}^n$ with $|a|=k(2g-2+n)$. Let $s\neq t\in\{1,\dots,n\}$ be indices such that k does not divide a_s or a_t. Then, we have the following relation:*

$$\begin{aligned}&(a_s\psi_s - a_t\psi_t)\cdot[\overline{\mathcal{M}}_g(a)]\\ &= \sum_{(\overline{\Gamma},I)\in \mathrm{LG}_1^2(g,a)} f_{s,t}(\overline{\Gamma})\frac{m(\overline{\Gamma})}{|\mathrm{Aut}(\overline{\Gamma})|}\cdot \zeta_{\Gamma *}\left([\overline{\mathcal{M}}_{g_0}(I(v_0))]\otimes [\overline{\mathcal{M}}_{g_{-1}}(I(v_{-1}))]\right)\end{aligned} \tag{18}$$

In order to prove this lemma we will work with the multi-scale compactification of $\mathcal{M}_g(a)$ introduced in Section 2.4.

Proof of Lemma 3.2. Recall that we denote by $p:\mathbb{P}\Xi_g(a)\to\overline{\mathcal{M}}_g(a)$ the morphism defined as the composition of the desingularization and the forgetful morphism to $\overline{\mathcal{M}}_g(a)$. By the projection formula, the class $\psi_s[\overline{\mathcal{M}}_g(a)]$ is equal to $p_*(p^*\psi_i)$. Thus we now study the intersection theory on $\mathbb{P}\Xi_g(a)$.

Using Proposition 2.8, we can write $(a_s\psi_s-\eta)$ or $(a_t\psi_t-\eta)$ on $\mathbb{P}\Xi_g(a)$ as a sum on irreducible components of the boundary divisors indexed by the graphs in $\mathrm{LG}_1(g,a)$ such that s or t respectively are adjacent to a level -1 vertex. Moreover, the coefficients of an irreducible divisor appearing in the expression of $(a_s\psi_s-\eta)$ or $(a_t\psi_t-\eta)$ are equal. Thus, we write $(a_s\psi_s-a_t\psi_t)$ as $(a_s\psi_s-\eta)-(a_t\psi_t-\eta)$ to express it as a sum over irreducible components of the boundary divisors indexed by $\mathrm{LG}_1(g,a)$, where s and t are adjacent to distinct levels.

If $\overline{\Gamma}\in\mathrm{LG}_1(g,a)$ is a level graph with at least 2 vertices of level 0, then the fibers of p restricted to the corresponding divisors have positive dimension. Thus such a graph contributes trivially to the expression of $(a_s\psi_s-a_t\psi_t)\cdot[\overline{\mathcal{M}}_g(a)]$.

Moreover, for all graphs $\overline{\Gamma}\in\mathrm{LG}_1(g,a)$ involved in the expression of $(a_s\psi_s - a_t\psi_t)$, either s or t is adjacent to this unique vertex of level 0. Thus the global residue condition defined is empty as k does not divide a_s, nor a_t (see Remark 2.7). Therefore, if such a graph has at least 2 vertices of level -1, then the fibers of p restricted to the corresponding divisor have positive dimension and once again such graphs contribute trivially to the expression of $(a_s\psi_s - a_t\psi_t) \cdot [\overline{\mathcal{M}}_g(a)]$.

Hence $(a_s\psi_s - a_t\psi_t)[\overline{\mathcal{M}}_g(a)]$ is expressed as a sum over level graphs in $\mathrm{LG}_1^2(g,a)$ and the coefficient for each such graph is exactly $f_{s,t}(\overline{\Gamma})m(\overline{\Gamma})/|\mathrm{Aut}(\overline{\Gamma})|$, again by Proposition 2.8 . □

3.2. **Proof of Proposition 3.1.** Finally, we can combine the results of Sections 2.2 and 3.1 to prove Proposition 3.1. As a first step, we observe that it suffices to show the proposition for particular input vectors a.

Lemma 3.3. *Let $g, n \geq 0$ such that $2g-2+n > 0$. Then Proposition 3.1 is true if and only if it is true for $k > 0$ and $a \in (\mathbb{Z} \setminus k\mathbb{Z})^n$.*

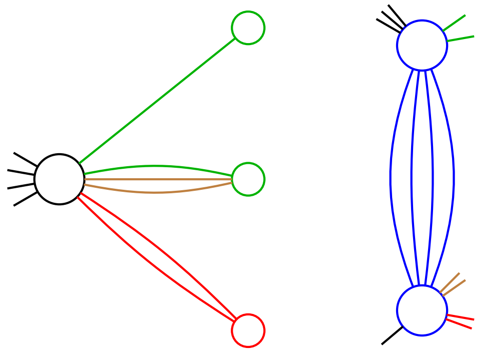
Proof. First note that for valid input vectors $a \in \Lambda$ contained in the sublattice $\Lambda \subset \mathbb{Z}^n$ of vectors whose sum is divisible by $2g-2+n$, the parameter $k = k(a)$ can be computed from a. Thus the statement in Proposition 3.1 is purely a statement about cycles depending on these input vectors. The crucial observation is that the two sides of the equality (17) in Proposition 3.1 are polynomial in the entries of a by [PZ] (for the left-hand side) and Lemma A.1 (for the right-hand side). Thus we conclude by observing that the statement of the proposition is vacuous for $n = 1$ and that for $n \geq 2$, the set of vectors $a \in \Lambda$ with $k = k(a) > 0$ and $a \in (\mathbb{Z} \setminus k\mathbb{Z})^n$ is Zariski-dense in $\mathbb{R}^n$ and so any polynomial equality satisfied for such a is satisfied everywhere. □

Remark 3.4. Note that for a level graph in $\mathrm{LG}_1^2(g,a)$, the level structure is uniquely determined by the underlying twisted graph (and the automorphisms of the level graph are automorphisms of the underlying twisted graph). Thus in the following proof, we will consider these objects as twisted graphs.

Proof of Proposition 3.1. By Lemma 3.3 it suffices to show equality (17) for vectors $a \in \mathbb{Z}^n$ summing to some integer multiple $k(2g-2+n)$ of $2g-2+n$ such that $a \in (\mathbb{Z} \setminus k\mathbb{Z})^n$. The overall strategy of our proof is as follows:

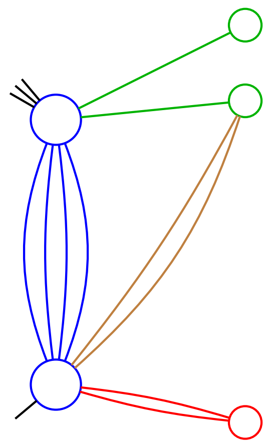
- Step 1 In the left-hand side of (17), we use Theorem 2.4 to replace the double ramification cycle by a sum over star graphs with strata of k-differentials and strata of 1-differentials at the vertices. By the assumption on a, all markings must go to the central vertex here.
- Step 2 Then we use the splitting formula from Lemma 3.2 on the central vertex (again made possible by the assumption on a) to replace it by a sum over 2-twisted graphs glued into this vertex. At this stage, we have expressed the left-hand side of (17) as a sum over graphs (20) with appropriate twists on all half-edges and strata of k- and 1-differentials in the vertices.
- Step 3 In the final step we interpret the top and bottom part of the graph (20) as simple star graphs and recombine the corresponding sub-summations into double ramification cycles using Theorem 2.4 in the opposite of the direction used before. We are left with a sum over 2-twisted graphs with double ramification cycles at the vertices, obtaining the right-hand side of (17).

The outline above contains all relevant mathematical ideas going into the proof, and a reader satisfied by this outline may safely skip to the next section. The remainder of the argument below will focus on controlling the combinatorics and multiplicities involved with the above manipulations of graph sums.



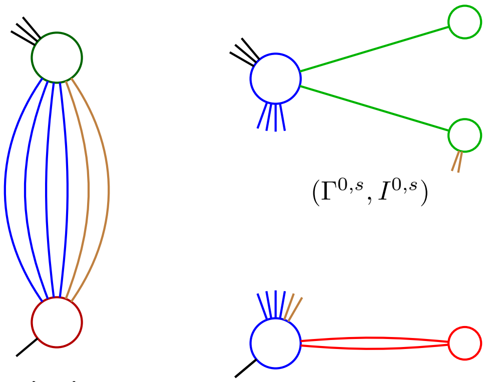
(Γ^s, I^s) (Γ^ℓ, I^ℓ)

(19)



(Γ, I)

(20)



$(\widehat{\Gamma}^\ell, \widehat{I}^\ell)$ $(\Gamma^{0,s}, I^{0,s})$ $(\Gamma^{-1,s}, I^{-1,s})$

(21)

Step 1: We start with the expression $(a_s\psi_s - a_t\psi_t)\mathrm{DR}_g(a)$. By the assumption that $a \in (\mathbb{Z} \setminus k\mathbb{Z})^n$, we can apply Theorem 2.4 to replace the double ramification cycle by a sum over star graphs (Γ^s, I^s) with strata of k-differentials/1-differentials at the central and outlying vertices (see the left side of (19)). Moreover, all (legs corresponding to) markings must be on the central vertex since all of their weights are in $\mathbb{Z} \setminus k\mathbb{Z}$ (see the definition of a simple star graph). In particular, the markings s, t belong to the same stratum $\overline{\mathcal{M}}_{g(v_0)}(I_s(v_0))$ of k-differentials that is glued into v_0 and the term $a_s\psi_s - a_t\psi_t$ can be pulled back to that vertex.
Step 2: Again using the assumption $a \in (\mathbb{Z} \setminus k\mathbb{Z})^n$ combined with the fact that all entries of $I_s(v_0)$ are negative, we can then apply Lemma 3.2 to replace the expression

$$(a_s\psi_s - a_t\psi_t) \cdot \overline{\mathcal{M}}_{g(v_0)}(a, I_s(v_0))$$

on the central vertex by a sum over 2-twisted graphs (Γ^ℓ, I^ℓ) glued into this vertex (see the right side of (19)).

Overall, we have by now written $(a_s\psi_s - a_t\psi_t)\mathrm{DR}_g(a)$ as a sum indexed by simple star graphs (Γ^s, I^s) and 2-twisted graphs (Γ^ℓ, I^ℓ) gluable into the central vertex of Γ^s. The individual summands are described by the twisted graphs (Γ, I) depicted in (20) obtained by performing this gluing operation and, up to rational coefficients which we make more precise below, they are given by a pushforward under ξ_Γ of strata of k-differentials and 1-differentials at the two central, respectively the outlying vertices of Γ. Let us make a couple of observations at this point:

- Due to the factor $f_{s,t}(\Gamma^\ell, I^\ell)$ appearing in Lemma 3.2, any (Γ, I) appearing with nonzero coefficient will have to satisfy that s, t appear at two different vertices (the two central vertices depicted in blue). Using this observation, we see that we can uniquely[3] reconstruct (Γ^s, I^s) and (Γ^ℓ, I^ℓ) given (Γ, I): we obtain Γ^s by contracting all edges between the two central components, and we obtain Γ^ℓ by cutting all other edges and removing all other vertices.
- The outlying vertices of Γ (which are in natural bijection to the outlying vertices of Γ^s) can naturally be placed at the top or bottom of Γ (and coloured green and red). The rule is that any vertex connecting *only* to the bottom central vertex will be at the bottom (red) and all others will be at the top (green). This is motivated by the fact that the twist I at all half-edges at the outlying vertices is positive, and thus in the partial order on $V(\Gamma)$ coming from the twist I they should be strictly above any other vertex they connect to.
- We color all the edges of Γ by
  - blue for the edges between the two central vertices,
  - green and red for the edges on the top and bottom level between central and outlying vertices,
  - brown for the edges connecting the lower central vertex of Γ with an upper outlying vertex.

Step 3: Finally, we regroup the terms in the sum over (Γ, I) to obtain the right-hand side of (17). For this, given (Γ, I) we compute a triple of twisted graphs as follows:

---

[3]Strictly speaking the unique reconstruction requires the additional data of an identification of the edges of Γ^s with the edges in Γ incident to the non-central vertices. We will suppress this detail for now and return to it during the last part of the proof, when we match the precise coefficients and multiplicities of all involved terms.

- We obtain a 2-twisted graph $(\widehat{\Gamma}^\ell, \widehat{I}^\ell)$ by *contracting* all red and green edges (and preserving the brown edges).
- We obtain simple star graphs $(\Gamma^{i,s}, I^{i,s})$, $i = 0, -1$, by *cutting* all blue and brown edges and taking the connected components of the top and bottom levels.

From the construction, it is clear that (Γ, I) can be reconstructed by gluing the graphs $(\Gamma^{i,s}, I^{i,s})$ into the vertices of $(\widehat{\Gamma}^\ell, \widehat{I}^\ell)$, so that again Step 3 defined a combinatorial bijection.

The graphs $(\widehat{\Gamma}^\ell, \widehat{I}^\ell)$ appearing like this are precisely the indices of the sum on the right-hand side of (17). On the other hand, the double ramification cycles appearing in this right-hand side can be replaced by sums over star graphs by Theorem 2.4. Indeed, since a_s, a_t are assumed to be not divisible by k and since s, t go to opposite sides of $\widehat{\Gamma}^\ell$, the assumptions of Theorem 2.4 are satisfied and the simple star graphs that appear are precisely of the form $(\Gamma^{i,s}, I^{i,s})$.

Comparison of coefficients: At this point we have described how to use Theorem 2.4 and Lemma 3.2 to expand the left-hand side of (17) into a graph sum (with insertions being strata of k-differentials and 1-differentials) and how to regroup this sum using Theorem 2.4 to obtain the right-hand side of (17). We established these expansions and regroupings on the levels of the involved combinatorial objects (i.e. twisted graphs), but it remains to be verified that in the final step, all summands appear with the correct rational coefficients. Looking at the relevant formulas (17), (8) and (18) most of the factors are easily matched:[4]

For the multiplicities $m(\Gamma, I)$ we see that the union of half-edges (and twists) of Γ^s, Γ^ℓ is in natural correspondence to the union of half-edges (and twists) of $(\widehat{\Gamma}^\ell, \widehat{I}^\ell)$ and the $(\Gamma^{i,s}, I^{i,s})$, and thus

$$m(\Gamma^s, I^s) \cdot m(\Gamma^\ell, I^\ell) = m(\widehat{\Gamma}^\ell, \widehat{I}^\ell) \cdot m(\Gamma^{0,s}, I^{0,s}) \cdot m(\Gamma^{-1,s}, I^{-1,s})\,. \tag{22}$$

Since the outlying vertices of Γ^s are the union of the outlying vertices of the $\Gamma^{i,s}$, we have

$$k^{|V_{\mathrm{Out}}(\Gamma^s)|} = k^{|V_{\mathrm{Out}}(\Gamma^{0,s})|} \cdot k^{|V_{\mathrm{Out}}(\Gamma^{-1,s})|}\,. \tag{23}$$

As the levels in $\Gamma^\ell, \widehat{\Gamma}^\ell$ to which the markings s, t go are the same, we also have

$$f_{s,t}(\Gamma^\ell, I^\ell) = f_{s,t}(\widehat{\Gamma}^\ell, \widehat{I}^\ell)\,. \tag{24}$$

It remains to match the automorphism factors. As might be expected, this is the most tricky part of the argument, and the naive equality of products of sizes of automorphism groups in fact fails. The reason is that implicit in Steps 1 to 3 we have assumed e.g. an identification between half-edges at some vertex and the legs of a graph that is glued into this vertex.

To make precise statements here, it is advantageous to reformulate the graph sums in (17), (8) and (18) in terms of sums over twisted graphs (Γ, I) together with a bijective numbering $o : E(\Gamma) \to [[1, \ldots, e]]$ of all involved edges, where $e = |E(\Gamma')|$. The symmetric group S_e operates transitively on all such numberings and the stabilizer of an isomorphism class of a numbered graph is precisely the

[4]Instead of performing Step 3 in the direction described above (regrouping terms (Γ, I)), the reader might find it easier to go in the opposite direction and expand the right-hand side of (17) into a graph sum. With this interpretation, the equations (22),(23), (24), (25) below show how the coefficient of the graph sum term (Γ, I) from Steps 1,2 agrees with the coefficient obtained from this inverse of Step 3.

automorphism group of the underlying twisted graph. Using the Orbit-Stabilizer Theorem from group theory one checks that formulas (17), (8) and (18) remain valid under replacing

- the sums over isomorphism classes of (Γ, I) satisfying the respective properties with sums over isomorphism classes of $(\Gamma, I, o : E(\Gamma) \xrightarrow{\sim} [[1, \ldots, e]])$,
- replacing the factors $1/|\mathrm{Aut}(\Gamma, I)|$ by $1/e!$.

With this insight, we can essentially finish the argument. To set notation, let $e_{\mathrm{bl}}, e_{\mathrm{br}}, e_{\mathrm{g}}, e_{\mathrm{r}}$ be the numbers of blue, brown, green and red edges in the pictures (19), (20) and (21). Then passing to the numbered version of formulas (17), (8) and (18) has the following effects on the coefficients of (Γ, I) in (20):

- In (19) we can assume to have a numbering on the $e_{\mathrm{g}} + e_{\mathrm{br}} + e_{\mathrm{r}}$ edges of Γ^s and legs of Γ^ℓ and that the gluing respects this ordering. Similarly we have a numbering on the e_{bl} edges of Γ^ℓ. Overall, the glued graph (Γ, I) appears with a coefficient of
$$\frac{1}{(e_{\mathrm{g}} + e_{\mathrm{br}} + e_{\mathrm{r}})! e_{\mathrm{bl}}!}$$
and inherits two numberings (on the green, brown and red edges and on the blue edges).
- In (21) we can have a numbering on the $e_{\mathrm{bl}} + e_{\mathrm{br}}$ edges of $\widehat{\Gamma}^\ell$ and legs of $\Gamma^{i,s}$ with the gluing respecting this ordering. Similarly we have a numbering on the e_{g} edges of $\Gamma^{0,s}$ and the e_{r} edges of $\Gamma^{-1,s}$. Overall, the glued graph (Γ, I) appears with a coefficient of
$$\frac{1}{(e_{\mathrm{bl}} + e_{\mathrm{br}})! e_{\mathrm{g}}! e_{\mathrm{r}}!}$$
and inherits three numberings (on the blue and brown, the green and the red edges).

To conclude we must not just compare the coefficients mentioned above but also with how many different numberings each (Γ, I) can appear. To do this, we use the following result:
Fact : Given sets $M_1, \ldots, M_u$ of sizes $m_1, \ldots, m_u$, the map from the set of orderings of $M = M_1 \sqcup \cdots \sqcup M_u$ to the product of the set of orderings on each M_i taking the induced order on the subsets $M_i \subset M$ has fibres of size
$$\binom{m_1 + \ldots + m_u}{m_1, \ldots, m_u}.$$

Thus given a graph (Γ, I) with four orderings on its edges (one for each colour), there are $\binom{e_{\mathrm{g}}+e_{\mathrm{br}}+e_{\mathrm{r}}}{e_{\mathrm{g}}, e_{\mathrm{br}}, e_{\mathrm{r}}}$ orderings in (19) inducing the given four orders, and $\binom{e_{\mathrm{bl}}+e_{\mathrm{br}}}{e_{\mathrm{bl}}, e_{\mathrm{br}}}$ orderings from (21). Overall, the desired equality of the coefficients in (17) then follows from the identity

$$\frac{1}{(e_{\mathrm{g}} + e_{\mathrm{br}} + e_{\mathrm{r}})! e_{\mathrm{bl}}!} \binom{e_{\mathrm{g}} + e_{\mathrm{br}} + e_{\mathrm{r}}}{e_{\mathrm{g}}, e_{\mathrm{br}}, e_{\mathrm{r}}} = \frac{1}{(e_{\mathrm{bl}} + e_{\mathrm{br}})! e_{\mathrm{g}}! e_{\mathrm{r}}!} \binom{e_{\mathrm{bl}} + e_{\mathrm{br}}}{e_{\mathrm{bl}}, e_{\mathrm{br}}}. \tag{25}$$

□

4. Identities satisfied by $\mathcal{A}_g$

In this section we use Proposition 3.1 to determine three identities satisfied by the functions $\mathcal{A}_g$. We show that these identities determine the functions $\mathcal{A}_g$ to prove Theorem 1.1.

Lemma 4.1. *The functions $\mathcal{A}_g = \mathcal{A}_g(a_1, \ldots, a_n)$ are polynomials in the entries a_i, of total degree at most $2g$ and symmetric in the arguments $a_2, \ldots, a_n$. Moreover, they satisfy the following identities:*

$$\mathcal{A}_g(a_1, \ldots, a_n, k) = \mathcal{A}_g(a_1, \ldots, a_n) \quad \text{for } k = \frac{a_1 + \ldots + a_n}{2g-2+n}, \tag{26}$$

$$a_1 \cdot \mathcal{A}_g(a_1, \ldots, a_n, 0) + \sum_{i>1} (a_i - k)\mathcal{A}_g(\ldots, a_i - k, \ldots) = \frac{1}{2}\sum_{j=0}^{k} j(k-j) \cdot \mathcal{A}_{g-1}(a_1, \ldots, a_n, -j, j-k) \tag{27}$$

for $k = \dfrac{a_1 + \ldots + a_n}{2g-2+n+1}$,

$$\mathcal{A}_g|_{|a|=0} = [z^{2g}] \frac{\prod_{i=1}^n \mathcal{S}(a_i z)}{\mathcal{S}(z)}. \tag{28}$$

Proof. The polynomiality and degree bound for $\mathcal{A}_g$ follow from [PZ] and the symmetry in $a_2, \ldots, a_n$ follows from the S_n-equivariance of the double ramification cycle in its entries and the resulting S_{n-1}-invariance of the definition (2) of $\mathcal{A}_g$.

The identity (26) follows from the fact that the class $\mathrm{DR}_g(a_1, \ldots, a_n, k)$ is the pull-back of $\mathrm{DR}_g(a_1, \ldots, a_n)$ under the forgetful morphism of the last marking (this follows from Invariance II of [BHP+20]). The identity (28) was proved in [BSSZ15, Theorem 1]. Thus it remains to prove the identity (27). We rewrite:

$$a_1 \mathcal{A}_g(a_1, \ldots, a_n, 0) = \int_{\overline{\mathcal{M}}_{g,n+1}} \psi_1^{2g-3+n} \cdot (a_1\,\psi_1 - 0 \cdot \psi_{n+1}) \mathrm{DR}_g(a_1, \ldots, a_n, 0).$$

Using Proposition 3.1, we can express the right hand side as a sum on graphs with 2 vertices, with markings 1 and $n+1$ being on different vertices. For dimension reasons, the only graphs contributing non-trivially are the graphs such that the vertex that does not contain 1 (and hence *does* contain $n+1$) is of genus 0 and has exactly 3 half-edges. It occurs in 2 family of cases:

- either the $(n+1)$st marking is on a vertex with another marking $i \neq 1$, connected to the other vertex by 1 edge;
- or the $(n+1)$st marking is on a vertex with no other marking, connected to the other vertex by 2 edges.

Keeping only these two contributions gives the second relation. □

Next, we prove that the properties above are sufficient to characterize $\mathcal{A}_g$.

Proposition 4.2. *All functions $\mathcal{A}_g$ are uniquely determined by $\mathcal{A}_0$, $\mathcal{A}_1$, and the properties in Lemma 4.1.*

The proof of the previous Proposition uses in a substantial way the polynomiality and symmetry properties of $\mathcal{A}_g$. Before we begin the proof, we need some technical lemmas concerning such symmetric polynomial functions.

Lemma 4.3. *Let $g : \mathbb{Z}^n \to \mathbb{Q}^n$ be a polynomial function of total degree (at most) D in the variables $y_1, \dots, y_n$, which is symmetric in $y_2, \dots, y_n$. Then there exist coefficients*

$$c_{d,\mu} \in \mathbb{Q} \text{ for } 0 \le d \le D \text{ and } \mu = (m_1, \dots, m_\ell) \text{ a partition of size at most } D - d$$

such that

$$g(y) = \sum_{d,\mu} c_{d,\mu} \cdot y_1^d e_\mu(y) \tag{29}$$

where $e_\mu(y)$ is the elementary symmetric polynomial defined by:

$$\prod_{i=1}^n (X + y_i) = \sum_{i=0}^n e_{n-i}(y) X^i$$
$$e_\mu = \prod_{i=1}^\ell e_{m_i}.$$

For $n > D$, the coefficients $c_{d,\mu}$ in the representation (29) are unique. In other words, the functions $y_1^d e_\mu(y)$ form a basis of the space of all functions g as above.

Furthermore, recalling that $e_m(y_1, \dots, y_n) = 0$ for $n < m$, the function g with representation (29) satisfies

$$g(y_1, \dots, y_{n-1}, 0) = \sum_{d,\mu} c_{d,\mu} \cdot y_1^d e_\mu(y_1, \dots, y_{n-1}) \,. \tag{30}$$

Proof. The set of functions g as above forms the degree at most D part of the ring

$$Q = \mathbb{Q}[y_1] \otimes_{\mathbb{Q}} \mathbb{Q}[y_2, \dots, y_n]^{S_{n-1}} \,,$$

which, by the fundamental theorem of symmetric polynomials, has a canonical basis given by $y_1^d e_\mu(y_2, \dots, y_n)$ where all parts of μ are of size at most $n-1$. We want to show that the system $\{y_1^d e_\mu(y) : d, \mu\}$ is a generating set of Q, for $y = (y_1, \dots, y_n)$. Since the system is closed under multiplication and contains y_1, it suffices to show that all functions $g = e_m(y_2, \dots, y_n)$ have a representation (29). This easily follows by induction on m using the fact that

$$e_m(y) - e_m(y_2, \dots, y_n) = y_1 \cdot h \text{ for } h \in Q \text{ of degree at most } m - 1 \,.$$

For $n > D$ we have that automatically all partitions μ of size at most D have that all parts are bounded by $n-1$. Thus the functions $y_1^d e_\mu(y_2, \dots, y_n)$ with $d + |\mu| \le D$ form a basis of $Q_{\le D}$. But since the number of such pairs d, μ equals the number of elements $y_1^d e_\mu(y)$ and since these generate $Q_{\le D}$ by the first part of the proof, they form a basis as desired. The last statement (30) of the lemma is immediate from the definition of the elementary symmetric polynomials. □

Lemma 4.4. *Let $n \ge 1$, $\ell \ge 0$ and $\mu = (m_1, \dots, m_\ell)$ a partition. Let furthermore $y_1, \dots, y_n, k$ be formal variables. Then we have*

$$e_\mu(y_1, \dots, y_n, -k) = e_\mu + O(k) \,, \tag{31}$$

$$\sum_{i=1}^n y_i e_\mu(y_1, \dots, y_i - k, \dots, y_n) = e_1 e_\mu - |\mu| k e_\mu + O(k^2) \,, \tag{32}$$

where on the right-hand sides $e_1 = e_1(y)$*,* $e_\mu = e_\mu(y)$*.*[5]

Proof. The statement (31) follows immediately from the fact that

$$e_m(y_1, \dots, y_n, -k) = e_m(y_1, \dots, y_n) - k e_{m-1}(y_1, \dots, y_n) = e_m + O(k)\,.$$

For the second property, we expand $e_\mu = \prod_{i=1}^{\ell} e_{m_j}$ and approximate the result up to first order in k:

$$\begin{aligned}
&\sum_{i=1}^n y_i \prod_{j=1}^{\ell} e_{m_j}(y_1, \dots, y_i - k, \dots, y_n)\\
=&\sum_{i=1}^n y_i \prod_{j=1}^{\ell} \left(e_{m_j}(y) - k e_{m_j-1}(y_1, \dots, \widehat{y_i}, \dots, y_n)\right)\\
=&\sum_{i=1}^n y_i \left(e_\mu(y) - k \sum_{j=1}^{\ell} e_{\widehat{\mu}^j} e_{m_j-1}(y_1, \dots, \widehat{y_i}, \dots, y_n)\right) + O(k^2)
\end{aligned}$$

where in the last line we use the notation $\widehat{\mu}^j = (m_1, \dots, \widehat{m_j}, \dots, m_\ell)$ for the partition obtained from μ by removing the j-th part. Pulling the sum over i we obtain

$$\begin{aligned}
&\sum_{i=1}^n y_i \left(e_\mu(y) - k \sum_{j=1}^{\ell} e_{\widehat{\mu}^j} e_{m_j-1}(y_1, \dots, \widehat{y_i}, \dots, y_n)\right) + O(k^2)\\
=&e_1 e_\mu - k \sum_{j=1}^{\ell} e_{\widehat{\mu}^j} \sum_{i=1}^n y_i e_{m_j-1}(y_1, \dots, \widehat{y_i}, \dots, y_n) + O(k^2)\\
=&e_1 e_\mu - k \sum_{j=1}^{\ell} e_{\widehat{\mu}^j} m_j e_{m_j} + O(k^2)\\
=&e_1 e_\mu - k|\mu| e_\mu + O(k^2)\,,
\end{aligned}$$

which finishes the proof. Here in the second to last step we used that

$$\sum_{i=1}^n y_i e_{m-1}(y_1, \dots, \widehat{y_i}, \dots, y_n) = m e_m(y)\,.$$

And indeed, the two sides of this last equality are both sums of terms $y_{i_1} \cdots y_{i_m}$ for subsets $\{i_1, \dots, i_m\} \subseteq \{1, \dots, n\}$ of size m, and each such subset appears precisely m times (e.g. on the left for each choice of $i = i_u$, $u = 1, \dots, m$). □

Proof of Proposition 4.2. For $g > 1$ we denote by $\widetilde{\mathcal{A}}_g : \mathbb{Z}^n \to \mathbb{Q}$ the function defined by

$$\widetilde{\mathcal{A}}_g(y_1, \dots, y_n) = \mathcal{A}_g(y_1 + k, \dots, y_n + k) \text{ for } k = \frac{\sum_i y_i}{2g-2}.$$

The functions $\mathcal{A}_g$ and $\widetilde{\mathcal{A}}_g$ determine each other and so it suffices to show that $\widetilde{\mathcal{A}}_g$ is determined by the analog of the properties from Lemma 4.1 together with the data in genus $0, 1$.

We continue to fix $g > 1$ and assume for the moment that $n > 2g$. Then since $\widetilde{\mathcal{A}}_g$ is a polynomial in its entries, of degree at most $2g$ and symmetric in the last $n-1$

[5]Here the notations $O(k)$, $O(k^2)$ stand for the sets of elements of the polynomial ring $\mathbb{Q}[y_1, \dots, y_n, k]$ divisible by k, k^2, respectively.

arguments, we know from Lemma 4.3 that there exist *unique* coefficients $c_{g,d,\mu} \in \mathbb{Q}$ for all $d \geq 0$, and $\mu = (m_1, \ldots, m_\ell)$ a partition of size $|\mu|$ at most $2g - d$, such that

$$\widetilde{\mathcal{A}}_g(y) = \sum_{g,\mu} c_{g,d,\mu} \cdot y_1^d \, e_\mu(y) \, . \tag{33}$$

Moreover, the first identity (26) of Lemma 4.1 implies that $\widetilde{\mathcal{A}}_g(y_1, \ldots, y_n, 0) = \widetilde{\mathcal{A}}_g(y_1, \ldots, y_n)$. Therefore, by the last part of Lemma 4.3 the coefficients $c_{g,d,\mu}$ are not only unique but in fact *independent of* n, since the corresponding expression (33) restricts correctly to all lower numbers of marked points.

Now consider the third property (28). Note that the operation of restricting $\widetilde{\mathcal{A}}_g$ to the locus where $e_1(y) = y_1 + \ldots + y_n$ vanishes precisely corresponds to setting $e_1(y) = 0$ in the expression (33). Given a partition μ, we denote by $M(\mu)$ its number of entries equal to 1. Then the third identity (28) determines all coefficients $c_{g,d,\mu}$, with $M(\mu) = 0$. Therefore, we will show that the second identity (27) provides a relation allowing to compute the remaining coefficients, inductively on g, d, and $M(\mu)$.

Fix now $g > 1$ and $n > 2g + 1$. Then the second relation (27) translates into the conventions of $\widetilde{\mathcal{A}}_g$ as[6]

$$\begin{aligned}(y_1 + k) \cdot \widetilde{\mathcal{A}}_g(y_1, \ldots, y_n, -k) \; &+ \sum_{i>1} y_i \widetilde{\mathcal{A}}_g(y_1, \ldots, y_i - k, \ldots, y_n) \\ &= \frac{1}{2} \sum_{j=0}^{k} j(k-j) \cdot \widetilde{\mathcal{A}}_{g-1}(y_1, \ldots, y_n, -j-k, j-2k)\end{aligned} \tag{34}$$

for any integers $y_1, \ldots, y_n$ and where $k = (y_1 + \ldots + y_n)/(2g - 1)$. Indeed, the reader can verify that applying the definition of $\widetilde{\mathcal{A}}$ to equation (34) and substituting $y_i = a_i - k$ precisely gives the second relation (27). Note that the formula for k was specifically chosen in such a way that

$$\frac{y_1 + \ldots + y_n - k}{2g - 2} = k \, ,$$

so that the value of k computed from the arguments of the functions $\widetilde{\mathcal{A}}_g$ and $\widetilde{\mathcal{A}}_{g-1}$ in (34) agrees with the defining formula of k.

Both the left and right hand side are polynomials in

$$\mathbb{Q}[y_1] \otimes \mathbb{Q}[y_2, \ldots, y_n]^{S_{n-1}} \tag{35}$$

of degree at most $2g + 1$. Thus by Lemma 4.3 and the assumption $n > 2g + 1$ both sides are unique linear combinations of the functions $y_1^d e_\mu(y)$. Comparing coefficients, we obtain a system of relations between the coefficients $c_{g,d,\mu}$ and $c_{g-1,d,\mu}$ of the functions $\widetilde{\mathcal{A}}_g$ and $\widetilde{\mathcal{A}}_{g-1}$ which appear. We will argue that this system uniquely determines the coefficients $c_{g,d,\mu}$ assuming the values of the coefficients $c_{g-1,d,\mu}$ are known by induction. In particular, since the right-hand side of equation

[6] There is a small annoying point: for $g = 2$ the term $\widetilde{\mathcal{A}}_{g-1}$ appearing below is not defined, since in genus 1 the value of k cannot be reconstructed from the values of the y_i. To fix the argument that follows, one can just replace the evaluation of $\widetilde{\mathcal{A}}_{g-1}$ on the right by the expression $\mathcal{A}_1(y_1+k, \ldots, y_n+k, -j, j-k)$ and the entire argument works verbatim. Indeed, the only thing we need is that this term can be written as a linear combination of $y_1^d e_\mu(y)$ and that by assumption we know the coefficients of the linear combination in genus 1.

(34) is assumed to be known (and thus forms the inhomogeneous part of the linear system between the $c_{g,d,\mu}$), we can focus on the left-hand side of the equation.

To control the notation below, we declare that for A, B, C elements of the ring (35), the equation $A = B+(C)$ means that $A-B$ is contained in the ideal generated by C. We begin with some preparatory computations: recall that $k = e_1/(2g-1)$. Given any partition μ, we write it as $\mu = (1^{M(\mu)}) + \mu'$, so that μ' contains no parts 1. Then by Lemma 4.4 and using that $(k) = (e_1)$ as ideals in (35), we have

$$\begin{aligned} e_{\mu'}(y_1, \dots, y_n, -k) &= e_{\mu'} + (e_1)\,, \\ \sum_{i\geq 1} y_i e_{\mu'}(y_1, \dots, y_i - k, \dots, y_n) &= \left(1 - \frac{|\mu'|}{2g-1}\right) e_1 e_{\mu'} + (e_1^2)\,. \end{aligned}$$

Now we put back the power $e_1^{M(\mu)}$ in the two equations above. For this observe that

$$e_1(y_1, \dots, y_n, -k) = e_1(y_1, \dots, y_i - k, \dots, y_n) = e_1 - k = \frac{2g-2}{2g-1} e_1\,.$$

Multiplying each summand on the left hand side above with $e_1^{M(\mu)}$, we obtain

$$\begin{aligned} e_{\mu}(y_1, \dots, y_n, -k) &= \left(\frac{2g-2}{2g-1}\right)^{M(\mu)} e_\mu + (e_1^{M(\mu)+1})\,, \\ \sum_{i\geq 1} y_i e_{\mu}(y_1, \dots, y_i - k, \dots, y_n) &= \left(\frac{2g-2}{2g-1}\right)^{M(\mu)} \left(1 - \frac{|\mu'|}{2g-1}\right) e_1 e_\mu + e_1^{M(\mu)+2}. \end{aligned}$$

Now we start putting together the terms appearing in the left-hand side of (34). For this observe that

$$\begin{aligned} &(y_1 + k) e_\mu(y, -k) + \sum_{i>1} y_i e_\mu(\dots, y_i - k, \dots) \\ =&(y_1 + k) e_\mu(y, -k) - y_1 e_\mu(y_1 - k, \dots) + \sum_{i\geq 1} y_i e_\mu(\dots, y_i - k, \dots) \\ =& \left(y_1 + \frac{e_1}{2g-1}\right) \cdot \left(\left(\frac{2g-2}{2g-1}\right)^{M(\mu)} e_\mu + (e_1^{M(\mu)+1}) \right) - (y_1) \\ &+ \left(\frac{2g-2}{2g-1}\right)^{M(\mu)} \left(1 - \frac{|\mu'|}{2g-1}\right) e_1 e_\mu + (e_1^{M(\mu)+2}) \\ =& \left(\frac{2g-2}{2g-1}\right)^{M(\mu)} \cdot \left(\frac{1}{2g-1} + 1 - \frac{|\mu'|}{2g-1}\right) e_1 e_\mu + (y_1) + (e_1^{M(\mu)+2})\,. \end{aligned}$$

Multiplying the two sides of the above equality by y_1^d, we precisely get the terms on the left-hand side of (34) that are multiplied by the coefficient $c_{g,d,\mu}$. Thus we see that $c_{g,d,\mu}$ appears as a factor in front of a term $y_1^{\widetilde{d}} e_{\widetilde{\mu}}$ only if either

- $\widetilde{d} = d$ and $\widetilde{\mu} = (1) + \mu$, in which case it appears with the coefficient

$$\left(\frac{2g-2}{2g-1}\right)^{M(\mu)} \cdot \frac{2g - |\mu'|}{2g-1}\,, \tag{36}$$

- or possibly when $\widetilde{d} > d$ or $M(\widetilde{\mu}) > M(\mu) + 1$.

To conclude, we first remark that the coefficient (36) can only vanish when $|\mu'| = 2g$. Since the total degree of $\widetilde{\mathcal{A}}_g$ is bounded by $2g$, this can only happen when $\mu = \mu'$ and thus $M(\mu) = 0$, in which case we already know the coefficient $c_{g,d,\mu}$ by the third assumption (28). Thus we can assume that (36) does not vanish and looking at the total factor in front of $y_1^d e_1 e_\mu$ on the left-hand side of (34), we see a nonzero multiple of $c_{g,d,\mu}$ as well as some multiples of coefficients $c_{g,d'',\mu''}$ satisfying $d'' < d$ or $M(\mu'') < M(\mu)$. By doing an induction on d and $M(\mu)$, which we can start with the cases $d = -1$, $M(\mu) = -1$ which vanish, we can assume that we have already computed these lower coefficients. Thus we can uniquely determine $c_{g,d,\mu}$ as claimed above, finishing the proof. □

Proof of Theorem 1.1. We denote by

$$\begin{aligned} B_g : \mathbb{Z}^n &\to \mathbb{Q} \\ a &\mapsto [z^{2g}]\ \exp\left(\frac{a_1 z \cdot \mathcal{S}'(kz)}{\mathcal{S}(kz)}\right) \frac{\prod_{i>1} \mathcal{S}(a_i z)}{\mathcal{S}(z)\mathcal{S}(kz)^{2g-1+n}} \end{aligned}$$

Using Proposition 4.2, we will prove prove Theorem 1.1 by showing that B_g is a polynomial of degree $2g$ that satisfies the four identites of Lemma 4.1, and that $\mathcal{A}_0 = B_0$ and $\mathcal{A}_1 = B_1$.

The polynomiality and degree bound of B_g follows by the rules of expansion of power series: the variable z appears in the definition of B_g always with coefficients that are either constant or linear in the a_i. Thus after substituting into power series and taking products of these, the coefficient of z^{2g} is indeed a polynomial of degree at most $2g$. The symmetry of the formula in the arguments $a_2, \ldots, a_n$ is likewise obvious.

The fact that these numbers satisfy the first identity (26) is straightforward, the third identity follows the fact that setting $|a| = 0$ is equivalent to setting $k = 0$ and the fact that $\mathcal{S}(0) = 1, \mathcal{S}'(0) = 0$.

In genus 0 we have that $\mathrm{DR}_0(a) = [\overline{\mathcal{M}}_{0,n}]$ is independent of a and so

$$\mathcal{A}_0(a) = \int_{\overline{\mathcal{M}}_{0,n}} \psi_1^{n-3} = 1\,,$$

e.g. by [BSSZ15, Theorem 1], which agrees with $B_0 = 1$.

In genus 1, we can reduce the check that $\mathcal{A}_1 = B_1$ to the case $k = 0$. Indeed, the fact that $\omega_C \cong \mathcal{O}_C$ for smooth genus 1 curves C can be used to show that the double ramification cycle is independent of k, in the sense that

$$\mathrm{DR}_1(a_1, \ldots, a_n) = \mathrm{DR}_1(a_1 - k, \ldots, a_n - k)\,.$$

Alternatively, this equality can be checked directly from the formula of $\mathrm{DR}_1(a)$ by applying known relations between divisor classes on $\overline{\mathcal{M}}_{1,n}$. In any case we can conclude that

$$\mathcal{A}_1(a_1, \ldots, a_n) = \mathcal{A}_1(a_1 - k, \ldots, a_n - k)\,. \tag{37}$$

Note here that the arguments on the right-hand side of (37) indeed sum to zero. As for the function B_1, an explicit computation using the substition $a_1 = kn - \sum_{i>1} a_i$

and the expansion $\mathcal{S}(u) = 1 + u^2/24 + O(u^4)$ shows that

$$B_1(a) = [z^2]\exp\left(\frac{a_1 z \cdot \mathcal{S}'(kz)}{\mathcal{S}(kz)}\right)\frac{\prod_{i>1}\mathcal{S}(a_i z)}{\mathcal{S}(z)\mathcal{S}(kz)^{n+1}}$$
$$= \frac{1}{24}\left(-1 + \sum_{i=2}^{n}(a_i - k)^2\right).$$

From this form we then immediately see that

$$B_1(a_1, \ldots, a_n) = B_1(a_1 - k, \ldots, a_n - k)\,. \tag{38}$$

Combining equations (37) and (38) we see that the functions $\mathcal{A}_1$ and B_1 are determined (via the same transformation) by their values on vectors a with $|a| = 0$. But for such vectors, the equality $\mathcal{A}_1(a) = B_1(a)$ follows from (28).

Thus, we only need to show that the numbers $B_g(a)$ satisfy the identity (27). We fix a vector $a \in \mathbb{Z}^n$, and we introduce the following formal series in z:

$$B_g(a)(z) = \exp\left(\frac{a_1 z \cdot \mathcal{S}'(kz)}{\mathcal{S}(kz)}\right)\cdot\frac{\prod_{i>1}\mathcal{S}(a_i z)}{\mathcal{S}(z)\mathcal{S}(kz)^{2g-2+n}}.$$

The derivative of this power series is given by:

$$\frac{d}{dz}B_g(a)(z) = B_g(a)(z)\cdot\left(\frac{a_1}{kz} - \frac{a_1}{kz\mathcal{S}(kz)^2} - (k(2g-2+n) - a_1)\frac{\mathcal{S}'(kz)}{\mathcal{S}(kz)}\right.$$
$$\left. - \frac{\mathcal{S}'(z)}{\mathcal{S}(z)} + \sum_{i>1} a_i \frac{\mathcal{S}'(a_i z)}{\mathcal{S}(a_i z)}\right).$$

On the other hand, we write the terms involved identity (27) using coefficient extraction formulas:

$$a_1 B_g(a,0) = a_1[z^{2g}]\mathcal{S}(kz)^{-2}B_g(a)(z),$$
$$\sum_{i>1}(a_i - k)B_g(\ldots, a_i - k, \ldots) = [z^{2g}]B_g(a)(z)\Bigg((2g - a_1/k)k$$
$$+ (k(2g-1+n) - a_1)kz\frac{\mathcal{S}'(kz)}{\mathcal{S}(kz)} - k\sum_{i>1}\frac{a_i z\mathcal{S}'(a_i z)}{\mathcal{S}(a_i z)}\Bigg),$$
$$\frac{1}{2}\sum_{0<j<k} j(k-j)B_{g-1}(a, -j, -k+j) = k[z^{2g}]B_g(a)(z)\left(kz\frac{\mathcal{S}'(kz)}{\mathcal{S}(kz)} - z\frac{\mathcal{S}'(z)}{\mathcal{S}(z)}\right),$$

The last expression follows from the formula

$$\sum_{0<j<k}\frac{j(k-j)}{2}\mathcal{S}(jz)\mathcal{S}((k-j)z) = \frac{1}{z^2}\left(k\cosh(kz/2) - \frac{\sinh(kz/2)\cosh(z/2)}{\sinh(z/2)}\right), \tag{39}$$

which can be proven by inserting the definition of $\mathcal{S}$, multiplying both sides with $\sinh(z/2)$, expanding everything in terms of exponential functions and comparing coefficients of each terms $e^{cz/2}$ for $c \in \mathbb{Z}$.

In order to show that B_g satisfies identity (27), we will use the fact that

$$-k[z^{2g}]z^{2g+1}\frac{d}{dz}z^{-2g}B_g(a)(z) = 0,$$

for all values of a, since the z^{-1}-coefficient of the derivative of any Laurent series vanishes. Using the above expression of the derivative of $B_g(a)(z)$, we obtain:

$$\begin{aligned}
&-k[z^{2g}]z^{2g+1}\ \frac{d}{dz}z^{-2g}B_g(a)(z)\\
&\quad= B_g(a)(z)\cdot\Bigg((2g-a_1/k)k+\frac{a_1}{\mathcal{S}(kz)^2}+(k^2(2g-2+n)-a_1k)\frac{z\mathcal{S}'(kz)}{\mathcal{S}(kz)}\\
&\qquad+\frac{kz\mathcal{S}'(z)}{\mathcal{S}(z)}-\sum_{i>1}a_ikz\frac{\mathcal{S}'(a_iz)}{\mathcal{S}(a_iz)}\Bigg).\\
&\quad= a_1B_g(a,0)-\frac{1}{2}\sum_{0<j<k}j(k-j)B_{g-1}(a,-j,-k+j)\\
&\qquad+\sum_{i>1}(a_j-k)B_g(\dots,a_j-k,\dots),
\end{aligned}$$

thus showing that $B_g(a)$ satisfy the identity (27) for all values of a. □

5. Top-ψ for spin components

In this section we analyze integrals of ψ-classes on spin components. In order to prove Theorem 1.4, we will follow the same strategy as for Theorem 1.1. That is, we will first show that the intersection of ψ-classes with strata of $[\overline{\mathcal{M}}_g(a)]^{\mathrm{spin}}$ may be expressed in terms of classes supported on graphs with 2 vertices. These formulas hold for odd values of k and a, but we will use the polynomiality of the function $\mathcal{A}_g^{\mathrm{spin}}$ implied by Assumption 1.3 to find constraints on this function that determine it uniquely (in the spirit of Lemma 4.1 and Proposition 4.2 in the non-spin case).

5.1. **Parity of boundary divisors.** In this section we work over the moduli space of multi-scale differentials $\mathbb{P}\Xi_g(a)$ (see Section 2.4) and explain how spin components intersect some special boundary divisors.

Recall that for an odd k and an odd signature a, the parity of a k-differential $(C,[\omega],x_1,\dots,x_n)\in\mathbb{P}\Omega_g(a)$ is defined as the parity of the spin structure

$$\omega_C^{\otimes(-k+1)/2}\otimes\mathcal{O}_C\left(\frac{a_1-k}{2}x_1+\ldots+\frac{a_n-k}{2}x_n\right).$$

We will denote the parity of the the k-differential $(C,[\omega],x_1,\dots,x_n)\in\mathbb{P}\Omega_g(a)$ by $\Phi(C,\omega)$.

First of all we describe the boundary divisors with precisely one edge appearing in the spin components. These vertical level graphs parametrize equivalence classes of multi-scale k-differentials $((C_i,[\eta_i])_{i=0,-1},[\sigma])$ where σ is a global prong matching. Recall that the equivalence relation is given by the action of the level rotation torus.

Proposition 5.1. *Let $\mathbb{P}\Xi(\overline{\Gamma})$ be a boundary divisor of $\mathbb{P}\Xi_g(a)$ with only one vertical edge connecting two vertices, for a odd, and let $((C_i,[\eta_i])_{i=0,-1},[\sigma])$ be a point of this divisor with stable graph $\overline{\Gamma}$. Let $(C,[\omega])$ be a nearby smoothing. Then $\Phi(C,\omega)=\Phi(C_0,\eta_0)+\Phi(C_{-1},\eta_{-1})$.*

Proof. Consider a one-parameter family $\varphi: B\to\mathbb{P}\Xi_g(a)$ smoothing the given multi-scale differential $((C_i,[\eta_i])_i)=\varphi(b_0)$ and let $\pi:\mathcal{C}\to B$ be the associated family of curves over B. Consider the modified family $\widetilde{\mathcal{C}}$ obtained by blowing up the nodal point of the fibre over b_0, giving an exceptional divisor $E \subset \widetilde{\mathcal{C}}$. Then on $\widetilde{\mathcal{C}}$ consider the line bundle

$$\mathcal{L}_B = \omega_\pi^{\otimes \frac{-k+1}{2}} \otimes \mathcal{O}_{\mathcal{C}}\left(\frac{a_1 - k}{2} x_1 + \ldots + \frac{a_n - k}{2} x_n + \frac{m(\overline{\Gamma}) - 1}{2} \cdot E + m(\overline{\Gamma}) \cdot C_{-1}\right),$$

where $C_{-1} \subseteq \widetilde{\mathcal{C}}_{b_0}$ is the component of the special fibre $\widetilde{\mathcal{C}}_{b_0}$ associated to the level -1 vertex of $\overline{\Gamma}$ and $m(\overline{\Gamma})$ is equal to the unique twist on the edge of $\overline{\Gamma}$. Note that it follows from our assumptions that $m(\overline{\Gamma})$ is odd, so that $(m(\overline{\Gamma}) - 1)/2$ is indeed an integer. Then from the definition of $\mathbb{P}\Xi_g(a)$ one checks there exists a natural map $\mathcal{L}_B^{\otimes 2} \to \omega_\pi$ giving a family of spin structures on $\widetilde{\mathcal{C}}$ (in the sense of [Cor89]). But now it follows from the results of [Cor89] that the parity of this spin structure

- is constant in the family,
- at the special fibre $\widetilde{\mathcal{C}}_{b_0} = C_0 \cup E \cup C_{-1}$ is equal to the sum of the parities of the spin structures on the irreducible components C_0, C_{-1} modulo 2 ([Cor89, Example 6.1]).

Comparing the parity at the special fibre with the one at the nearby smoothing $(C, [\omega])$ concludes the proof. □

While we wrote the proof above for graphs with a single edge, a similar argument in fact works for arbitrary graphs of compact type, and thus we obtain the following generalization.

Proposition 5.2. *Let $\mathbb{P}\Xi(\overline{\Gamma})$ be a boundary stratum of $\mathbb{P}\Xi_g(a)$ such that $\overline{\Gamma}$ is a tree and all edges of $\overline{\Gamma}$ are vertical, for a odd. Let $((C_i, [\eta_i])_{v \in V(\overline{\Gamma})}, [\sigma])$ be a point of this divisor with stable graph $\overline{\Gamma}$. Let $(C, [\omega])$ be a nearby smoothing. Then*

$$\Phi(C, \omega) = \sum_{v \in V(\Gamma)} \Phi(C_v, \eta_v).$$

We now present a lemma about the parity of banana divisors, i.e. a divisor whose associated level graph has two vertical edges and two vertices. This result was never explicitly stated but it was implicitly used in [EMZ03, Section 14.1] (see also [CG21] for related arguments). Note that if a is odd, then the twists at the two nodes have to be of the same parity.

Lemma 5.3. *Let k, a be odd and let $\mathbb{P}\Xi(\overline{\Gamma})$ be a banana boundary divisor of $\mathbb{P}\Xi_g(a)$, parametrizing multi-scale differentials of the form $((C_i, [\eta_i])_{i=0,-1}, [\sigma])$. Let $(C, [\omega])$ be a nearby smoothing.*

(1) If $\overline{\Gamma}$ has odd twists, then $\Phi(C, \omega) = \Phi(C_0, \eta_0) + \Phi(C_{-1}, \eta_{-1})$.
(2) If $\overline{\Gamma}$ has even twists, then $\Phi(C, \omega) = 0$ for half of the prong matchings equivalence classes and $\Phi(C, \omega) = 1$ for the other half.

We prove this lemma in 2 steps: first we show that it holds in the case $k = 1$, and then use this special case to show it in full generality.

Proof. Case of $k = 1$. We start by considering the case of $k = 1$. Recall that in the case of abelian differentials, the parity of a spin structure may also be defined in the following way using the Arf invariant (see [EMZ03] or [KZ03]). Consider a flat surface (C, ω) and fix a symplectic basis of homology $\{A_i, B_i\}_{i=\ldots,g}$. When the zeroes of the abelian differential ω have only even orders, we can define the parity of the spin structure via

$$\Phi(C,\omega) = \sum_{i=1}^{g} (\mathrm{Ind}_\omega(A_i)+1)(\mathrm{Ind}_\omega(B_i)+1) \pmod 2,$$

where $\mathrm{Ind}_\omega(\alpha)$ is the index of the tangent vector field of a closed curve α with respect to the flat metric induced by ω. One can show that this parity is independent from the choices.

Note that the plumbing construction described in [BCG+19b] is only local near the nodes, so we can choose a symplectic basis of homology on C_1 and C_2 and complete this basis to a symplectic basis of a smoothing C only by adding the vanishing cycles and their symplectic duals.

Given the previous remark, it is clear that in the case of only one vertical edge, the parity of (C,ω) is given by the sum of the parities of (C_i,η_i), since there are no vanishing cycles in this situation.

In the case of two nodes and two vertical edges, we can consider the symplectic basis on C given by the union of the symplectic bases on the C_i together with a vanishing cycle v and its symplectic dual u. We hence find

$$\Phi(C,\omega) = \Phi(C_0,\eta_0) + \Phi(C_{-1},\eta_{-1}) + (\mathrm{Ind}_\omega(v)+1)(\mathrm{Ind}_\omega(u)+1) \pmod 2.$$

Since a is of odd signature, the twists at the nodes (which are the same as the numbers of prongs) have to be of the same parity, which we denote by κ. From the definition of plumbing fixture of [BCG+19b], it is clear that the parity of the index $\mathrm{Ind}_\omega(v)$ of the vanishing cycle equals κ.

If κ is odd, then $\mathrm{Ind}_\omega(v)+1$ is even, and so we get that the parity of (C,ω) is the sum of the parities of the two levels. If κ is even, then the parity of (C,ω) is determined by the parity of the index $\mathrm{Ind}_\omega(u)$ of the dual cycle. The key observation now is that the parity of $\mathrm{Ind}_\omega(u)$ is changed whenever we move from an equivalence class of prong matchings to the next one. Indeed, if $[\sigma]$ is an equivalence class of prong matchings, then twisting σ by 2π at one node while keeping the other node fixed gives a different equivalence class, since the level rotation torus acts by twisting the prong matchings at the two nodes at the same time. Moreover, this twisting clearly alters the parity of $\mathrm{Ind}_\omega(u)$ by 1. Hence we find that for half of the prong matchings equivalence classes the number $\mathrm{Ind}_\omega(u)$ is of even parity, and for the other half it is of odd parity, so we finally obtain also the last claim.

Case of $k>1$. By degeneration arguments, we will reduce the proposition for arbitrary g to the case of genus 1. In the case of $g=1$ both the definition of strata of multi-scale differentials and parity is independent of k, so in this case the main statement for $k>1$ holds from the first step of the proof.

Consider a general banana boundary divisor $\mathbb{P}\Xi(\overline{\Gamma})$ in the case of $k>1$, and its level strata $\mathbb{P}\Xi(\overline{\Gamma})^{[i]}$, for $i=0,-1$. If the dimension of the level strata $\mathbb{P}\Xi(\overline{\Gamma})^{[i]}$ is positive, it is possible to find a divisor in $\mathbb{P}\Xi(\overline{\Gamma})^{[i]}$ with only one vertical edge and with a genus 0 top or bottom level stratum, respectively. By constructing the corresponding degeneration of $\mathbb{P}\Xi(\overline{\Gamma})$ via the clutching operation, we obtain a boundary component $\mathbb{P}\Xi(\overline{\Delta})$ of $\mathbb{P}\Xi_g(a)$ of codimension 3 such that its corresponding level graph $\overline{\Delta}$ is the one illustrated in Figure 1. Consider finally the codimension 2 boundary stratum $\mathbb{P}\Xi(\overline{\Lambda})$ given by the undegeneration of $\mathbb{P}\Xi(\overline{\Delta})$ obtained by collapsing the middle banana level passage. In particular the middle level $\mathbb{P}\Xi(\overline{\Lambda})^{[-1]}$

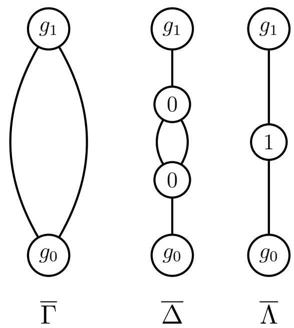


FIGURE 1. The graphs $\overline{\Gamma}, \overline{\Lambda}$ and their common degeneration $\overline{\Delta}$ appearing in the proof of Lemma 5.3.

is of genus 1 and there is only one vertical edge connecting it to the other two different levels.

By Proposition 5.1, we can write $\Phi(C, \omega) = \sum_{i=0}^{-2} \Phi(C_i^{\overline{\Lambda}}, \eta_i^{\overline{\Lambda}})$, where we denoted by $(C_i^{\overline{\Lambda}}, \eta_i^{\overline{\Lambda}})$ a point in $\mathbb{P}\Xi(\overline{\Lambda})$ in a neighborhood of $\mathbb{P}\Xi(\overline{\Delta}) = \mathbb{P}\Xi(\overline{\Gamma}) \cap \mathbb{P}\Xi(\overline{\Lambda})$. Using again Proposition 5.1, we also know that $\Phi(C_0^{\overline{\Gamma}}, \eta_0^{\overline{\Gamma}}) = \Phi(C_0^{\overline{\Lambda}}, \eta_0^{\overline{\Lambda}}) + \Phi(C_{-1}^{\overline{\Delta}}, \eta_{-1}^{\overline{\Delta}})$ and that $\Phi(C_{-1}^{\overline{\Gamma}}, \eta_{-1}^{\overline{\Gamma}}) = \Phi(C_{-2}^{\overline{\Lambda}}, \eta_{-2}^{\overline{\Lambda}}) + \Phi(C_{-2}^{\overline{\Delta}}, \eta_{-2}^{\overline{\Delta}})$. Since the prong matching equivalence classes of $\mathbb{P}\Xi(\overline{\Delta})$ correspond bijectively to the ones of $\mathbb{P}\Xi(\overline{\Gamma})$, we can apply the main statement for the genus 1 stratum $\mathbb{P}\Xi(\overline{\Lambda})^{[-1]}$ and its divisor yielding the degeneration $\mathbb{P}\Xi(\overline{\Delta})$. This, together with the previous equations, gives us the main statement for the banana divisors in the general case of $k > 1$, when the dimension of the level strata $\mathbb{P}\Xi(\overline{\Gamma})^{[i]}$ are positive.

If only one of the level strata $\mathbb{P}\Xi(\overline{\Gamma})^{[i]}$ is positive dimensional, the analogous proof also works by considering only the degeneration of the positive dimensional level stratum. If both level strata have dimension zero, then the level graph has genus one, so the first part of this proof applies. □

We say that $\mathbb{P}\Xi(\overline{\Gamma})$ is a multi-banana boundary divisor if $\overline{\Gamma}$ is a 2-level graph with only two vertices, one for each of the two levels, and at least two non-horizontal edges. We can now generalize Lemma 5.3 to the case of such graphs $\overline{\Gamma}$.

Proposition 5.4. *Let k, a be odd and let $\mathbb{P}\Xi(\overline{\Gamma})$ be a multi-banana boundary divisor of $\mathbb{P}\Xi_g(a)$, parametrizing multi-scale differentials of the form $((C_i, [\eta_i])_{i=0,-1}, [\sigma])$. Let $(C, [\omega])$ be a nearby smoothing.*

1. *If $\overline{\Gamma}$ has only odd twists, then $\Phi(C, \omega) = \Phi(C_0, \eta_0) + \Phi(C_{-1}, \eta_{-1})$.*
2. *If $\overline{\Gamma}$ has at least one even twist, then $\Phi(C, \omega) = 0$ for half of the prong matchings equivalence classes and $\Phi(C, \omega) = 1$ for the other half.*

Proof. First of all note that if a multi-banana level graph $\overline{\Gamma}$ has an odd, resp. even, number of edges, then the number of odd twists of $\overline{\Gamma}$ is odd, resp. even.

In a similar fashion as in the previous proof, we want to argue by using a degeneration and an inductive argument. By Lemma 5.3, the statement is true in the case of two edges. So we now assume the induction hypothesis that the statement is true for multibanana boundary components with one edge less than $\overline{\Gamma}$.

Consider first the case where $\overline{\Gamma}$ has at least one odd twist. We will treat afterwards the case where all twists are even (in particular this means that $\overline{\Gamma}$ has an even number of edges).

First consider a boundary component $\mathbb{P}\Xi(\overline{\Delta})$ of codimension two given by the degeneration of $\mathbb{P}\Xi(\overline{\Gamma})$ with level graph $\overline{\Delta}$ (illustrated in Figure 2) consisting of a triangle-like 3-level graph, where there is a long edge corresponding to an edge of $\overline{\Gamma}$ with odd prong a, only one edge between level -1 and level -2 (which has to have an odd prong d), and with the remaining edges of $\overline{\Gamma}$ corresponding to the edges from level 0 to level -1. This is obtained by the clutching operation of the degeneration of the bottom level $\mathbb{P}\Xi(\overline{\Gamma})^{[-1]}$ with a one edge 2-level graph $\overline{\Delta_{\overline{\Gamma}}} \in \mathrm{LG}_1(\mathbb{P}\Xi(\overline{\Gamma})^{[-1]})$. Consider finally the banana divisor $\mathbb{P}\Xi(\overline{\Lambda})$ with odd prongs such that $\mathbb{P}\Xi(\overline{\Delta}) = \mathbb{P}\Xi(\overline{\Gamma}) \cap \mathbb{P}\Xi(\overline{\Lambda})$. Then $\mathbb{P}\Xi(\overline{\Delta})$ is given by the clutching operation of a multi-banana boundary divisor $\overline{\Delta_{\overline{\Lambda}}} \in \mathrm{LG}_1(\mathbb{P}\Xi(\overline{\Lambda})^0)$ with one edge less than $\overline{\Gamma}$.

Consider first the subcase where $\overline{\Gamma}$ has at least an even twist. Then $\overline{\Delta_{\overline{\Lambda}}}$ has also an even twist, and by induction hypothesis half of the prong matchings of $\mathbb{P}\Xi(\overline{\Delta_{\overline{\Lambda}}})$ belong to one spin component and half to the other one. Moreover, again by induction hypothesis, we know that all the prong matchings of the banana divisor $\mathbb{P}\Xi(\overline{\Lambda})$ go to the same spin component. After the clutching operation, this implies that half of the prong matchings of $\mathbb{P}\Xi(\overline{\Delta})$ go to one component and half to the other. In order to get the desired result for $\mathbb{P}\Xi(\overline{\Gamma})$, it is enough to note that since $\overline{\Delta_{\overline{\Gamma}}}$ has only one edge, by Proposition 5.1 all the prong matchings of $\overline{\Delta_{\overline{\Gamma}}}$ go to the same component. This means that after undoing the clutching operation, we find that half of the prong matchings of $\overline{\Gamma}$ go to one component and half of them go to the other component in this case.

Consider now the second subcase where $\overline{\Gamma}$ has only odd twists. In this case, since as before we can apply the induction hypothesis on $\overline{\Delta_{\overline{\Lambda}}}$ and by using Proposition 5.1 for $\overline{\Delta_{\overline{\Gamma}}}$, we know that $\Phi(C_0^{\overline{\Lambda}}, \eta_0^{\overline{\Lambda}}) = \Phi(C_{-1}^{\overline{\Delta}}, \eta_{-1}^{\overline{\Delta}}) + \Phi(C_0^{\overline{\Delta}}, \eta_0^{\overline{\Delta}})$ and $\Phi(C_{-1}^{\overline{\Gamma}}, \eta_{-1}^{\overline{\Gamma}}) = \Phi(C_{-2}^{\overline{\Delta}}, \eta_{-2}^{\overline{\Delta}}) + \Phi(C_{-1}^{\overline{\Delta}}, \eta_{-1}^{\overline{\Delta}})$. Using again the induction hypothesis, we finally also know that $\Phi(C,\omega) = \Phi(C_{-1}^{\overline{\Lambda}}, \eta_{-1}^{\overline{\Lambda}}) + \Phi(C_0^{\overline{\Lambda}}, \eta_0^{\overline{\Lambda}})$ where (C,ω) is a smoothing in a neighborhood of $\mathbb{P}\Xi(\overline{\Delta})$. Putting together the previous equations (together with the obvious equalities $\Phi(C_0^{\overline{\Gamma}}) = \Phi(C_0^{\overline{\Delta}})$ and $\Phi(C_{-1}^{\overline{\Lambda}}) = \Phi(C_{-2}^{\overline{\Delta}})$), we get the desired result that the parity of (C,ω) is the sum of the parities of the levels of $\mathbb{P}\Xi(\overline{\Gamma})$ in this case of all even prongs.

The last case left to be treated is the one where $\overline{\Gamma}$ has an even number of edges and all the twists are even. We can still consider the previous setting, but in this case half of the prong matchings of $\mathbb{P}\Xi(\overline{\Lambda})$ are in one spin component and half in the other one. More specifically,the proof of Lemma 5.3 implies that for each choice of a prong matching at the edge with twist a, there are half of the prong matchings at the edge with twist d that are in one spin component and half in the other. Since the level rotation torus acts level-wise, if we fix a prong matching at the long edge of $\overline{\Delta}$ with twist a, by induction hypothesis half of the prong matchings at the b-edges and at the d-edge are in one component and the other half in the other component. By undegenerating $\overline{\Delta}$ to $\overline{\Gamma}$, we can forget the prong matchings at the d-edge, and find again that if we fix a prong matching at the a edge of $\overline{\Gamma}$, half of the prong matchings at the b-edges are in one component and the other half in the other one. Hence we have proven the desired result also in this last case. $\square$

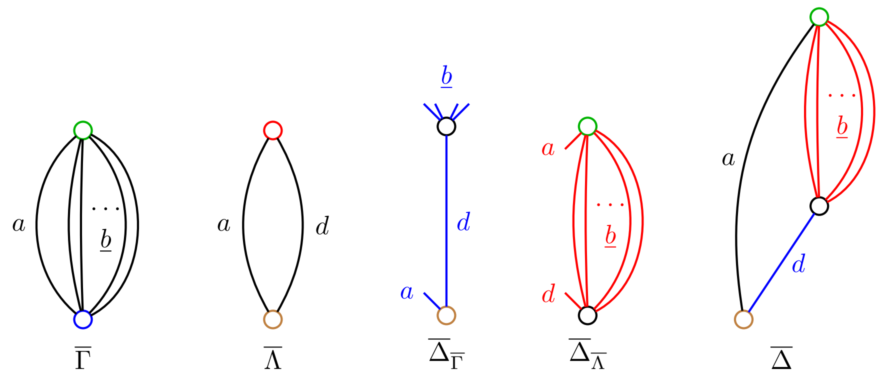


FIGURE 2. The multibanana graph $\overline{\Gamma}$ and the other graphs used in the degeneration argument.

5.2. **Splitting formulas with spin parity.** Using the result of the previous section, we prove a splitting formula for ψ-classes that takes into account the spin parity. Recall that $\mathrm{LG}_1^2(g,a)$ is the set of level graphs with exactly 2 vertices v_0 of level 0 and v_{-1} of level -1, and no horizontal edges.

Lemma 5.5. *We assume that k and the a_i's are all odd, and we chose $1 \le s < t \le n$ such that k does not divide s or t. Then, we have the following relation:*

$$(a_s\psi_s - a_t\psi_t)\cdot[\overline{\mathcal{M}}_g(a)]^{\mathrm{spin}} = \sum_{(\Gamma,I)\in \mathrm{LG}_1^2(g,a)} \beta(s,t,\Gamma,I),$$

where

$$\beta(s,t,\Gamma,I) \quad = \quad f_{s,t}(\Gamma,I)\frac{m(\Gamma,I)}{|\mathrm{Aut}(\Gamma,I)|}\zeta_{\Gamma*}\left([\overline{\mathcal{M}}_{g_0}(I(v_0))]^{\mathrm{spin}}\otimes[\overline{\mathcal{M}}_{g_{-1}}(I(v_{-1}))]^{\mathrm{spin}}\right)$$

if I has only odd entries, and $\beta(s,t,\Gamma,I)=0$ otherwise.

Proof. We consider the components $\mathbb{P}\Xi_g(a)^{\mathrm{odd}}$ or $\mathbb{P}\Xi_g(a)^{\mathrm{even}}$ of the moduli space of multi-scale differentials. Using the same arguments as in the non-spin case of Lemma 3.2, we can use Proposition 2.8 in order to write $(\eta + a_s\psi_s)$ and $(\eta + a_t\psi_t)$ on $\mathbb{P}\Xi_g(a)^{\mathrm{odd}}$ or $\mathbb{P}\Xi_g(a)^{\mathrm{even}}$ as a linear combination of components of the boundary divisors indexed by twisted graphs with 2 levels. As in the non-spin case, the contribution of a graph (Γ, I) vanishes if the graph is not in in $\mathrm{LG}_1^2(g,a)$.

Fix now a divisor $(\Gamma,I)\in \mathrm{LG}_1^2(g,a)$. Thanks to Proposition 5.1 and Proposition 5.4, if all entries of I are odd, then the parity of a connected component of this divisor is equal to the sum of the parities of the projections of this connected component in $\mathcal{M}_{g_0}(a_0,I_0)$ and $\mathcal{M}_{g_{-1}}(a_{-1},I_{-1})$. Thus the contribution of (Γ,I) to $(a_s\psi_s - a_t\psi_t)[\overline{\mathcal{M}}_g(a)]^{\mathrm{spin}}$ is given by:

$$f_{s,t}(\Gamma,I)\frac{m(\Gamma,I)}{|\mathrm{Aut}(\Gamma,I)|}\cdot\zeta_{\Gamma*}\left([\overline{\mathcal{M}}_{g_0}(a_0,I_0)]^{\mathrm{spin}}\otimes[\overline{\mathcal{M}}_{g_{-1}}(a_{-1},I_{-1})]^{\mathrm{spin}}\right).$$

If there is at least an even entry of I, then by Proposition 5.4 half of the prong matchings equivalence classes give a nearby even or odd differential. Since each choice of prong matchings equivalence classes contributes with the same amount to the difference of the psi-classes on each component, the contribution of (Γ, I) to $(a_s\psi_s - a_t\psi_t)[\overline{\mathcal{M}}_g(a)]^{\mathrm{spin}}$ is trivial as the even and odd contributions cancel out. □

Using Assumption 1.3 we denote by $\mathcal{A}_g^{\mathrm{spin}}$ the unique polynomial extending $\mathcal{A}_g^{\mathrm{spin}}$. We will prove the following refinement of the identities of Lemma 4.1 taking into account the spin parity.

Lemma 5.6. *For all even k, and all vectors a with even entries, we have:*

$$\mathcal{A}_g^{\mathrm{spin}}(a,k) = \mathcal{A}_g^{\mathrm{spin}}(a) \quad \text{for } k = \frac{a_1 + \ldots + a_n}{2g-2+n}, \tag{40}$$

$$a_1 \cdot \mathcal{A}_g^{\mathrm{spin}}(a,0) + \sum_{i>1}(a_i - k)\mathcal{A}_g^{\mathrm{spin}}(\ldots, a_i - k, \ldots) = \frac{1}{2}\sum_{\substack{j \text{ odd},\\ 0<j<k}} j(k-j)\mathcal{A}_{g-1}^{\mathrm{spin}}(a, -j, j-k) \tag{41}$$

$$\text{for } k = \frac{a_1 + \ldots + a_n}{2g-2+n+1},$$

$$\begin{aligned}(2g-2+n)a_1\mathcal{A}_g^{\mathrm{spin}}(a)_{|k=0} \\ = \quad & -\sum_{j>i>1}(a_i + a_j)\mathcal{A}_g^{\mathrm{spin}}(\ldots, \widehat{a_i}, \ldots, \widehat{a_j}, \ldots, a_i + a_j) \\ & -\frac{1}{2}\sum_{1<i\leq n}\sum_{\substack{j,\ell \text{ odd},\\ j\cdot\ell>0, j+\ell=a_i}} \mathrm{sign}(a_i)\cdot j\ell\cdot\mathcal{A}_{g-1}^{\mathrm{spin}}(\ldots, \widehat{a_i}, \ldots, a_n, j, \ell) + R(a)\,,\end{aligned} \tag{42}$$

where $R(a)$ is the evaluation of the polynomial expression

$$\frac{1}{6}\sum_{\substack{j,\ell,m \text{ odd},\\ \mathrm{sign}(j)=\mathrm{sign}(\ell)=\mathrm{sign}(m),\\ j+\ell+m=b}} j\ell m\cdot\mathcal{A}_{g-2}^{\mathrm{spin}}(a_1, \ldots, a_n, j, \ell, m)$$

in b at $b=0$.

Concering equation (42), it will turn out a posteriori from our formula that the term $R(a)$ vanishes. However, this is not obvious from the basic properties of $\mathcal{A}_g$, so for now we state the lemma in the slightly complicated version above.

Proof. First, we note that Lemma 5.5 is only valid for odd values of a and k. Moreover, by definition $\mathcal{A}_g^{\mathrm{spin}}(a) = \psi_1^{2g-3+n}\cdot[\overline{\mathcal{M}}_g(a)]^{\mathrm{spin}}$ if a_1 is either negative or not divisible by k. For all three identities above, the strategy is to prove the corresponding identity with these constraints and write their polynomial extension for even values of a.

Identity (40) follows immediately from the property $\pi^*\mathrm{DR}_g^{\mathrm{spin}}(a) = \mathrm{DR}_g^{\mathrm{spin}}(a,k)$ from Assumption 1.3.[7]

To prove identity (42), first note that the identity is trivial in case $n=1$: the restriction $k=0$ forces $a=(0)$, so that the left-hand side vanishes due to the factor

[7]Alternatively, it can be proven using only the polynomiality of $\mathcal{A}_g^{\mathrm{spin}}$ and property (3) from Assumption 1.3, since for a,k odd, the cycle $[\overline{\mathcal{M}}_g(a,k)]^{\mathrm{spin}}$ is the pull-back of $[\overline{\mathcal{M}}_g(a)]^{\mathrm{spin}}$. Note however that for a later part of the argument we need the full strength of part (2) of Assumption 1.3, and thus we cannot omit it.

$a_1 = 0$ and the right-hand side vanishes since the index-sets of the sum are empty. Thus we may assume $n \geq 2$.

To prove (42), we will first perform a calculation on a moduli space with one additional marking. Consider vectors $a' \in \mathbb{Z}^{n+1}$ with $|a'| = k(2g-1+n)$ so that both k and all entries of a' are odd and such that both a'_1 and a'_{n+1} are not divisible by k. Then we can use Lemma 5.5 above with $s = 1$, and $t = n+1$ to get the following expression:

$$(a'_1\psi_1 - a'_{n+1}\psi_{n+1})[\overline{\mathcal{M}}_g(a')]^{\mathrm{spin}} = \sum_{(\Gamma,I)} \beta(1, n+1, \Gamma, I). \tag{43}$$

When taking the intersection number of (43) with $\psi_{n+1}\psi_1^{2g-4+n}$, we note that on the left-hand side we can replace the cycle $[\overline{\mathcal{M}}_g(a')]^{\mathrm{spin}}$ by $\mathrm{DR}_g^{\mathrm{spin}}(a')$ by Assumption 1.3. On the other hand, for the right hand side almost all terms $\beta(1, n+1, \Gamma, I)$ contribute zero for dimension reasons. Going through all possible pairs (Γ, I), one sees that only four types of graphs allow nonzero intersection numbers:

- two vertices, connected by a single edge such that one vertex has genus zero and carries markings $i, j, n+1$ (for $1 < i < j < n+1$),
- two vertices, connected by two edges such that one vertex has genus zero and carries markings $i, n+1$ (for $1 < i \leq n$),
- two vertices, connected by three edges such that one vertex has genus zero and carries marking $n+1$,
- two vertices, connected by a single edge such that one vertex has genus one and carries marking $n+1$.

Using the explicit description of $\beta(1, n+1, \Gamma, I)$ given in Lemma 5.5, by pairing (43) with $\psi_{n+1}\psi_1^{2g-4+n}$ we get the following expression:

$$\begin{aligned} (44)\quad & \psi_{n+1}\psi_1^{2g-4+n}(a'_1\psi_1 - a'_{n+1}\psi_{n+1})\mathrm{DR}_g^{\mathrm{spin}}(a') \\ = \quad & -\sum_{j>i>1} (a'_i + a'_j + a'_{n+1} - 2k)\mathcal{A}_g^{\mathrm{spin}}(\ldots, \widehat{a'_i}, \ldots, \widehat{a'_j}, \ldots, a'_i + a'_j + a'_{n+1} - 2k) \\ & -\frac{1}{2}\sum_{1<i\leq n} \sum_{\substack{j,\ell \text{ odd},\\ j\cdot\ell>0, j+\ell=a'_i+a'_{n+1}-2k}} \mathrm{sign}(a'_i + a'_{n+1} - 2k)\cdot j\ell \cdot \mathcal{A}_{g-1}^{\mathrm{spin}}(\ldots, \widehat{a'_i}, \ldots, j, \ell) \\ & -(a'_{n+1} - 2k)\cdot \mathcal{A}_1^{\mathrm{spin}}(a'_{n+1}, 2k - a'_{n+1})\mathcal{A}_{g-1}^{\mathrm{spin}}(a'_1, \ldots, a'_n, -2k + a'_{n+1}) \\ & +R(a'), \end{aligned}$$

where

$$R(a') = \frac{1}{6} \sum_{\substack{j,\ell,m \text{ odd},\\ \mathrm{sign}(j)=\mathrm{sign}(\ell)=\mathrm{sign}(m),\\ j+\ell+m=a'_{n+1}-2k}} j\ell m \cdot \mathcal{A}_{g-2}^{\mathrm{spin}}(a'_1, \ldots, a'_n, j, \ell, m). \tag{45}$$

From Assumption 1.3 and Lemma A.4 it follows that all terms in the equality (44) except possibly $R(a')$ are polynomial on vectors a' such that all numbers $a'_i + a'_{n+1} - 2k$ are even, and of fixed signs, whereas $R(a')$ is only polynomial for vectors a' satisfying the slightly stronger condition that the entire vector a' is odd.

Thus, fixing a collection $\eta \in \{\pm 1\}^{n-1}$ of signs for the numbers $a'_i + a'_{n+1} - 2k$ cuts out a polyhedron in the space $\mathbb{R}^{n+1}$ of all vectors a' and we see that on the intersection Λ_η of that polyhedron with the lattice of vectors a' such that the numbers $a_i' + a_{n+1}' - 2k$ are even, the two sides of (44) are given by a polynomial in a', where $R(a')$ now really means the polynomial extending (45). Moreover, the vectors a' we considered above (k, a_i' odd and a_1', a_{n+1}' not divisible by k) are Zariski dense in each such domain, so that we have (44) as an equality on each Λ_η.

Now let us return to equality (42). Consider an even vector $a \in \mathbb{Z}^n$ with $|a| = 0$ and so that none of the a_i vanish. Again, since these are Zariski dense, it suffices to prove the statement for such vectors. Then we can consider (44) at the point $a' = (a_1, \ldots, a_n, 0)$, which satisfies $k = 0$ and lies in Λ_η for $\eta = (\mathrm{sign}(a_i))_{i=2}^n$. Then the right hand side of (44) simplifies to the right hand side of (42).

On the other hand, for the left-hand side we have

$$\begin{aligned} & a_1 \int_{\overline{\mathcal{M}}_{g,n+1}} \psi_{n+1} \psi_1^{2g-3+n} \mathrm{DR}_g^{\mathrm{spin}}(a_1, \ldots, a_n, 0) \\ =& a_1(2g-2+n) \int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{2g-3+n} \mathrm{DR}_g^{\mathrm{spin}}(a_1, \ldots, a_n) \\ =& a_1(2g-2+n) \mathcal{A}_g^{\mathrm{spin}}(a) \,, \end{aligned}$$

using the property $\pi^* \mathrm{DR}_g^{\mathrm{spin}}(a) = \mathrm{DR}_g^{\mathrm{spin}}(a, 0)$ for the forgetful morphism $\pi : \overline{\mathcal{M}}_{g,n+1} \to \overline{\mathcal{M}}_{g,n}$ from Assumption 1.3. This concludes the proof of equality (42).

The proof of identity (41) uses a similar strategy, now pairing equation (43) with ψ_1^{2g-3+n}, again for vectors a' such that both k and all a_i' are odd, the numbers a_1', a_{n+1}' are not divisible by k and additionally such that $a_1 < 0$.[8] The only two kinds of graphs that can contribute are

- two vertices, connected by a single edge such that one vertex has genus zero and carries markings $i, n+1$ (for $1 < i < j < n+1$),
- two vertices, connected by two edges such that one vertex has genus zero and carries marking $n + 1$.

We then obtain the equality

$$\begin{aligned} & a_1' \mathcal{A}_g^{\mathrm{spin}}(a') - a_{n+1}' \psi_{n+1} \psi_1^{2g-3+n} [\overline{\mathcal{M}}_g(a')]^{\mathrm{spin}} \\ = \quad & - \sum_{1 < i \le n} (a_i' + a_{n+1}' - k) \mathcal{A}_g^{\mathrm{spin}}(\ldots, a_i' + a_{n+1}' - k, \ldots) \\ & - \frac{1}{2} \sum_{\substack{j \text{ odd}, \\ 0 < j < k - a_{n+1}'}} j(k - a_{n+1}' - j) \cdot \mathcal{A}_{g-1}^{\mathrm{spin}}(a_1', \ldots, a_n', -j, j + a_{n+1}' - k) \,. \end{aligned}$$

Combining Assumption 1.3 with Lemma A.4 we see that both sides of this equality are polynomial in vectors a' for which $k - a_{n+1}'$ is even and that the equality holds on the level of these polynomials. Specializing to even k and even vectors $a' = (a_1, \ldots, a_n, 0)$ produces equality (41). □

Proposition 5.7. *The functions $\mathcal{A}_g^{\mathrm{spin}}$ are determined by $\mathcal{A}_0^{\mathrm{spin}}$, $\mathcal{A}_1^{\mathrm{spin}}$, and the identities of Lemma 5.6.*

Proof. The arguments used to prove Proposition 4.2 may be directly adapted to prove this Proposition. □

[8]The condition on a_1 being negative ensures that certain graphs with nontrivial residue conditions cannot appear below.

As for Theorem 1.1, the proof of Theorem 1.4 is obtained by a combination of Lemma 5.6 and Proposition 5.7 .

Proof of Theorem 1.4. We denote by

$$\begin{aligned} B_g^{\mathrm{spin}} : \mathbb{Z}^n &\to \mathbb{Q} \\ a &\mapsto 2^{-g}[z^{2g}]\exp\left(\frac{a_i z \cdot \mathcal{S}'(kz)}{\mathcal{S}(kz)}\right)\frac{\cosh(z/2)}{\mathcal{S}(z)}\frac{\prod_{i>1}\mathcal{S}(a_i z)}{\mathcal{S}(kz)^{2g-1+n}} \end{aligned}$$

the polynomial of degree $2g$ defined by the RHS of formula (5). By direct computation we have $\mathcal{A}_0^{\mathrm{spin}} = B_0^{\mathrm{spin}}$ and $\mathcal{A}_1^{\mathrm{spin}} = B_1^{\mathrm{spin}}$, using the identity $[\overline{\mathcal{M}}_0(a)]^{\mathrm{spin}} = [\overline{\mathcal{M}}_{0,n}]$ and the formula (13) for $[\overline{\mathcal{M}}_1(a)]^{\mathrm{spin}}$ discussed in Section 2.3.

Thus we will finish the proof by showing that the function B_g^{spin} satisfies the three relations of Lemma 5.6. By Proposition 5.7, this implies that $B_g^{\mathrm{spin}} = \mathcal{A}_g^{\mathrm{spin}}$ for all g. The fact that B_g^{spin} satisfies identity (40) is straightforward.

We first show that B_g^{spin} satisfy identity (42). If $k=0$, then the numbers $B_g^{\mathrm{spin}}(a)$ take the simpler form:

$$B_g^{\mathrm{spin}}(a) = 2^{-g}[z^{2g}]\frac{\cosh\left(\frac{z}{2}\right)}{\mathcal{S}(z)}\prod_{j\neq 1} S(a_j z)$$

The first thing we note in this expression, is that every variable $a_2, \dots, a_n$ has even degree in each monomial that appears. This implies that the term $R(a)$ in (42) vanishes by the last statement in Lemma A.4.

For the first term on the RHS of identity (42), we use addition formulas of the hyperbolic sine and the form of B_g^{spin} for $k=0$ to obtain:

$$\begin{aligned} &\sum_{j>i>1}(a_i+a_j)B_g^{\mathrm{spin}}(\dots,\widehat{a_i},\dots,\widehat{a_j},\dots,a_i+a_j) \\ = \quad & -2^{-g}[z^{2g}]\frac{\cosh\left(\frac{z}{2}\right)}{\mathcal{S}(z)}\sum_{i>1}(a_i+a_1)\cosh\left(\frac{a_i z}{2}\right)\prod_{j\neq i,1} S(a_j z)\,. \end{aligned}$$

For the second term, we use the identity

$$\mathrm{sign}(m)\sum_{\substack{j,\ell \text{ odd},\\ j\cdot\ell>0, j+\ell=m}} \frac{j\ell}{2}\mathcal{S}(jz)\mathcal{S}(\ell z) = \frac{1}{z^2}\left(\frac{m}{2}\cosh(mz/2) - \frac{\sinh(mz/2)}{\sinh(z)}\right), \tag{46}$$

which can be shown in a way analogous to the proof of identity (39). Then we have

$$\begin{aligned} &\frac{1}{2}\sum_{i>1}\sum_{\substack{j,\ell \text{ odd},\\ j\cdot\ell>0, j+\ell=a_i}} \mathrm{sign}(a_i)\cdot j\ell\cdot B_{g-1}^{\mathrm{spin}}(\dots,\widehat{a_i},\dots,j,\ell) \\ = \quad & 2^{-g+1}[z^{2g}]\frac{\cosh\left(\frac{z}{2}\right)}{\mathcal{S}(z)}\sum_{i>1}\left(\frac{a_i}{2}\cosh\left(\frac{a_i z}{2}\right) - \frac{\sinh\left(\frac{a_i z}{2}\right)}{\sinh(z)}\right)\prod_{j\neq 1,i}\mathcal{S}(a_j z) \end{aligned}$$

The sum of these two terms gives:

$$2^{-g}[z^{2g}]\frac{\cosh\left(\frac{z}{2}\right)}{\mathcal{S}(z)}\sum_{i>1}\left(\frac{-a_1}{2}\cosh\left(\frac{a_iz}{2}\right)-\frac{\sinh\left(\frac{a_iz}{2}\right)}{\sinh(z/2)\cosh(z/2)}\right)\prod_{j\neq 1,i}\mathcal{S}(a_jz)$$
$$=-2^{-g}a_1[z^{2g}]\left(z\frac{d}{dz}+n-2\right)\frac{\cosh\left(\frac{z}{2}\right)}{\mathcal{S}(z)}\prod_{i>1}\mathcal{S}(a_iz)$$
$$=-2^{-g}a_1(2g-2+n)[z^{2g}]\frac{\cosh\left(\frac{z}{2}\right)}{\mathcal{S}(z)}\prod_{i>1}\mathcal{S}(a_iz)$$

which is the opposite of the LHS of identity (42).

Finally, to check that the numbers $B_g^{\mathrm{spin}}(a)$ satisfy the identity (41) we proceed as in the non-spin case. For all $a\in\mathbb{Z}^n$, we by introduce the formal series:

$$B_g^{\mathrm{spin}}(a)(z) \quad = \quad 2^{-g}\exp\left(\frac{a_iz\cdot\mathcal{S}'(kz)}{\mathcal{S}(kz)}\right)\frac{\cosh(z/2)}{\mathcal{S}(z)}\frac{\prod_{i>1}\mathcal{S}(a_iz)}{\mathcal{S}(kz)^{2g-2+n}}.$$

Then we can check that:

$$-k[z^{2g}]\quad z^{2g+1}\frac{d}{dz}z^{-2g}B_g^{\mathrm{spin}}(a)(z)$$
$$= \quad a_1B_g^{\mathrm{spin}}(a,0)+\sum_{i>1}(a_j-k)B_g^{\mathrm{spin}}(\dots,a_j-k,\dots)$$
$$-\frac{1}{2}\sum_{\substack{j\text{ odd},\\ 0<j<k}} j(k-j)B_{g-1}^{\mathrm{spin}}(\dots,-j,j-k).$$

The left hand side vanishes, and the vanishing of the RHS shows that B_g satisfies identity 41. □

6. Euler characteristic of the minimal strata

In this section, we will work only under the assumption $k=1$. We will show how to obtain the formula of Euler characteristic of minimal strata and of the spin refinement of this Euler characteristic under Assumptions 1.3.

6.1. **Top-ψ for strata with residue conditions.** A central ingredient in our study will be spaces of meromorphic differentials on curves satisfying residue conditions at some of their poles and the intersection numbers of their fundamental classes with powers of ψ-classes. These spaces are very natural since they appear in the description of boundary strata of the usual spaces of abelian differentials (see e.g. [BCG+19b]). Moreover, their fundamental classes have many nice properties and explicit descriptions. For instance, as shown in [BRZ21] these classes form a partial cohomological field theory when requiring the residues to vanish at all of the poles, and for the case of differentials with precisely two zeroes there exists a connection to the KP hierarchy.

We start by choosing a vector $a=(a_0,\dots,a_n)$ such that $|a|=2g-1+n$, $a_0>0$ and $a_i<0$ for $1\le i\le n$. For $m\in\{1,\dots,n-1\}$, we denote by $\mathcal{M}_g^{\mathfrak{R}(m)}(a)\subset\mathcal{M}_g(a)$, the subset of the stratum of abelian differentials cut out by the condition that the residues $r_1,\dots,r_m$ at markings $1,\dots,n$ vanish. We denote by $\mathbb{P}\Xi_g^{\mathfrak{R}(m)}(a)$ its multi-scale differentials compactification as defined in [CMZ20a, Prop. 4.2]. In analogy with the previous sections, we define the intersection number

$$\mathcal{A}_g^{\mathfrak{R}(m)}(a) = \int_{\mathbb{P}\Xi^{\mathfrak{R}(m)}(a)} \psi_0^{2g+n-3-m}.$$

We use Proposition 2.9 which translates a stratum cut out by a residue condition as a tautological in class in the ambient stratum. We obtain a recursive formula in m, describing the previously defined intersection numbers.

Lemma 6.1. *Let $a = (a_0, a_1, \dots, a_n)$ be a vector such that $a_0 > 0$ and $a_i < 0$ for $i > 1$. For any $1 \leq m \leq n-1$, we have*

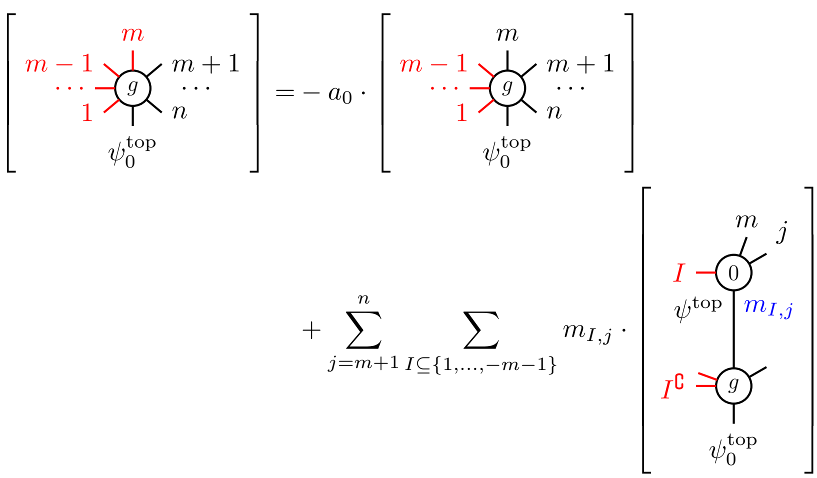
$$\mathcal{A}_g^{\mathfrak{R}(m)}(a) = -a_0 \mathcal{A}_g^{\mathfrak{R}(m-1)}(a) + \sum_{j=m+1}^{n} \sum_{I \subseteq \{1,\dots,m-1\}} m_{I,j} \cdot \mathcal{A}_0^{\mathfrak{R}(|I|)}(a_{I,j}) \cdot \mathcal{A}_g^{\mathfrak{R}(|I^{\complement}|)}(\hat{a}_{I,j}),$$

where $m_{I,j} = -a_m - \sum_{i \in I \cup \{j\}}(a_i - 1)$, and:

$$a_{I,j} = (m_{I,j}, \{a_i\}_{i \in I}, a_m, a_j),$$
$$\hat{a}_{I,j} = (a \setminus \{a_i\}_{i \in I \cup \{m,j\}}, -m_{I,j}).$$

We illustrate the lemma in equation (47) below. Here, a stable graph in brackets should be replaced by the product over the vertices v of the graph of values $\mathcal{A}_{g(v)}^{\mathfrak{R}(m(v))}(a(v))$, where $a(v)$ are the orders of zeros and poles at half-edges incident to v and $m(v)$ is the number of such half-edges with imposed residue conditions (which are drawn in red). We also write the order $m_{I,j}$ of the unique zero on the genus zero vertex in blue.

(47)



Proof. Let $1 \leq m \leq n-1$. We can use Propositions 2.8 and 2.9 in order to express the locus cut out by one residue condition $r_m = 0$ as:

$$-\eta \cdot [\mathbb{P}\Xi_g^{\mathfrak{R}(m-1)}(a)] = [\mathbb{P}\Xi_g^{\mathfrak{R}(m)}(a)] + \delta(m) \in A^1(\mathbb{P}\Xi_g^{\mathfrak{R}(m-1)}(a)),$$

such that $\delta(m)$ is a linear combination of the boundary divisors of $\mathbb{P}\Xi_g^{\mathfrak{R}(m-1)}(a)$, along which the residue r_m vanishes identically. Besides, we also have:

$$(-\eta + a_0\psi_0) \cdot [\mathbb{P}\Xi_g^{\mathfrak{R}(m-1)}(a)] = \delta \in A^1(\mathbb{P}\Xi_g^{\mathfrak{R}(m-1)}(a)),$$

where δ is the linear combination of all the boundary divisors $\mathbb{P}\Xi_g^{\mathfrak{R}(m-1)}(a)$ as a_0 is the only entry of a that is positive. Taking the difference of the two expressions implies:

$$[\mathbb{P}\Xi_g^{\mathfrak{R}(m)}(a)] = -a_0\psi_0 \cdot [\mathbb{P}\Xi_g^{\mathfrak{R}(m-1)}(a)] - \delta(m) + \delta.$$

Thus, the only divsors that are involved in $\delta(m) - \delta$ are the ones defined by level graphs in $\mathrm{LG}_1(g,a)$ such that the mth marking is supported on a vertex of level 0, and such that at least one of the markings in $\{m+1,\dots,n\}$ is attached to the same vertex.

Let $\overline{\Gamma}$ be such a level graph. Note that since marking 0 is the only zero of the differential, the graph has a unique vertex on level -1 and this vertex carries the leg associated to this marking. The intersection of the divisor associated to this graph with $\psi_0^{2g-3+n-m}$ vanishes, unless there is exactly one edge between one vertex of level 0 of genus 0 and one vertex of level -1 of genus g, and only one leg in $\{m+1,\dots,n\}$ is adjacent to the vertex of level 0. The coefficients $m_{i,j}$ are given by the conversion factor (16), in particular noting that these graphs have only the trivial automorphism. □

Note that apart from the theoretical argument given above, we were also able to check the formula from Lemma 6.1 in many non-trivial cases using the software package `diffstrata` [CMZ20b], which can compute formulas for the pushforwards of $\mathbb{P}\Xi_g^{\mathfrak{R}(m)}(a)$ to $\overline{\mathcal{M}}_{g,n+1}$ in terms of tautological classes.

We denote by $\mathrm{TR}(g,a)$ the set of genus g twisted graphs with legs a such that the vertex v_0 carrying the marking 0 has genus g. In particular, this forces the graph to be a tree, with all vertices apart from v_0 having genus 0. Such twisted graphs are uniquely determined by the underlying stable graph. Moreover there are no automorphisms of such objects as the poles are marked.

By a straightforward analysis, each vertex v has exactly one half-edge adjacent to it with a positive twist. We denote this number by $I(v)^+$. For such a twisted graph (Γ, I), we define

$$F(\Gamma, I) = \prod_{v\in V(\Gamma)} -\left(-I(v)^+\right)^{n(v)-2}, \tag{48}$$

where $n(v)$ is the number of half edges attached to v.

Proposition 6.2. *Let $a = (a_0, a_1, \dots, a_n)$ with $a_0 > 0$ and $a_i < 0$, for $i = 1, \dots, n$. Then, we have:*

$$\mathcal{A}_g^{\mathfrak{R}(n-1)}(a) = \sum_{(\Gamma,I)\in\mathrm{TR}(g,a)} -\mathcal{A}_g(I(v_0)) \cdot F(\Gamma, I).$$

Proof. The expression is obtained by applying Lemma 6.1 iteratively $n-1$ times to impose the vanishing of the residues at the $(n-1)$ first poles (and thus at all poles). Looking at equation (47), we see that when removing the residue condition at marking h incident to a vertex v, we obtain a sum of terms, corresponding to either

- leaving the graph unchanged, dropping the residue condition at h and obtaining a factor equal to minus the order of the unique zero on v, or
- sprouting off a genus zero vertex from v, carrying some markings I with residue conditions, the marking h and precisely one additional marking without residue condition. In this case, the multiplicity is given by the order of the unique zero on that new vertex.

Continuing to remove one residue condition at a time, we see terms with more complicated underlying stable graphs appearing, all of them trees with a unique genus g vertex, and thus contained in $\mathrm{TR}(g,a)$.

The terms appearing after $n-1$ steps are given by intersections numbers on strata of genus 0 differentials with residue constraints. For each of these strata we apply Lemma 6.1 until only the functions $\mathcal{A}_g$ and $\mathcal{A}_0$ are needed.

After $n-1$ steps, it can be shown that all twisted graphs in $\mathrm{TR}(g,a)$ appear once in the resulting formula with coefficient given by $-\mathcal{A}_g(I(v_0))\cdot F(\Gamma,I)$. This equality uses that after having removed all residue conditions, the contribution from the genus zero vertices is given by

$$\int_{\overline{\mathcal{M}}_{0,n}} \psi_1^{n-3} = 1 \qquad \square$$

The following proposition will not be used in the sequel, but it is interesting to note that in the genus 0 case, we can provide a closed formula.

Proposition 6.3. *We assume here that $g=0$. Let $a=(a_0,a_1,\dots,a_n)$ with $a_0>0$ and $a_i<0$ for $i=1,\dots,n$. For $n\geq 2$ and $0\leq m\leq n-2$, we have:*

$$\mathcal{A}_0^{\mathfrak{R}(m)}(a_0,\dots,a_n) = \frac{(n-2)!}{(n-m-2)!}\prod_{i=1}^{m}(-a_i)$$

Proof. We will show the stronger statement

$$[\mathbb{P}\Xi_0^{\mathfrak{R}(m)}(a)] = (-1)^m\prod_{i=1}^{m} a_i\psi_i. \tag{49}$$

which implies the statement above by the dilaton equation. The proof is similar to the proof of 6.1 above. We write:

$$\begin{aligned} -\eta\cdot[\mathbb{P}\Xi_0^{\mathfrak{R}(m-1)}(a)] &= [\mathbb{P}\Xi_0^{\mathfrak{R}(m)}(a)] + \delta(m)\\ (-\eta + a_m\psi_m)\cdot[\mathbb{P}\Xi_0^{\mathfrak{R}(m-1)}(a)] &= \delta(m). \end{aligned}$$

Indeed, in genus 0, the only boundary components along which the residue r_m vanishes identically are the ones such that the m-th marking belong to the level -1. These boundary components are also the only ones contributing to $(-\eta + a_m\psi_m)$ (and with the same coefficients). Thus

$$[\mathbb{P}\Xi_0^{\mathfrak{R}(m)}(a)] = -a_m\psi_m\cdot[\mathbb{P}\Xi_0^{\mathfrak{R}(m-1)}(a)],$$

and by induction we get the desired identity (49) $\square$

6.2. **Euler characteristic and minimal strata.** As before we consider $a=(a_0,a_1,\dots,a_n)$ with $a_0>0$ and $a_i<0$ for $i<0$, and we fix $0\leq m\leq n$. We will first work on the stratum $\mathbb{P}\Xi_g^{\mathfrak{R}(m)}(a)$ and denote by $d_g^m(a)$ its dimension. These strata are the ones that will appear as level -1 strata of boundary divisors of minimal holomorphic strata.

Let $\overline{\Gamma}$ be a level graph of depth L, and $-L\leq i\leq 0$. We denote by $d_{\overline{\Gamma}}^{[i]}$ the dimension of $\mathbb{P}\Xi(\overline{\Gamma})^{[i]}$ (the space of multi-scale differentials parametrizing the i-th level of this graph), and by $\eta_{\overline{\Gamma}}^{[i]}$ the first Chern class of its tautological line bundle. We will also denote by $\mathrm{LG}_L^{\mathfrak{R}(m)}(a)$ the set of level strata parametrizing boundary components of $\mathbb{P}\Xi_g^{\mathfrak{R}(m)}(a)$ (see [CMZ20a, Prop. 4.2]). Note that, for $\overline{\Gamma}\in$

$\mathrm{LG}_L^{\mathfrak{R}(m)}(a)$, the residue condition defining the level strata $\mathbb{P}\Xi(\overline{\Gamma})^{[i]}$ is a combination of the GRC of the graph and the condition $\mathfrak{R}(m)$.

Proposition 6.4. *We assume that $a_0 > 0$, and $a_i < 0$ for all $i \geq 1$. Then:*

$$\int_{\mathbb{P}\Xi_g^{\mathfrak{R}(m)}(a)} (a_0\psi_0)^{d_g^m(a)} = \sum_{L=0}^{d_g^m(a)} \sum_{\overline{\Gamma}\in \mathrm{LG}_L^{\mathfrak{R}(m)}(g,a)} \frac{m(\overline{\Gamma})}{|\mathrm{Aut}(\overline{\Gamma})|} \prod_{i=0}^{-L} \int_{\mathbb{P}\Xi(\overline{\Gamma})^{[i]}} (\eta_{\overline{\Gamma}}^{[i]})^{d_{\overline{\Gamma}}^{[i]}}$$

where $m(\overline{\Gamma})$ is the product of the twists at the edges of $\overline{\Gamma}$.

Proof. Since there is only one zero, by Proposition 2.8 and the conversion factor (16), we have

$$\eta = a_0\psi_0 - \sum_{\overline{\Gamma}\in \mathrm{LG}_1^{\mathfrak{R}(m)}(g,a)} \ell(\overline{\Gamma}) \cdot [\mathbb{P}\Xi(\overline{\Gamma})] \in A^1(\mathbb{P}\Xi_g^{\mathfrak{R}(m)}(a)).$$

Consider now the chain of equalities

$$\begin{aligned}
&\int_{\mathbb{P}\Xi_g^{\mathfrak{R}(m)}(a)} \eta^{d_g^m(a)} = \int_{\mathbb{P}\Xi_g^{\mathfrak{R}(m)}(a)} \eta^{d_g^m(a)-1} \left(a_0\psi_0 - \sum_{\overline{\Gamma}\in \mathrm{LG}_1^{\mathfrak{R}(m)}(g,a)} \ell(\overline{\Gamma})[\mathbb{P}\Xi(\overline{\Gamma})] \right) \\
&= \int_{\mathbb{P}\Xi_g^{\mathfrak{R}(m)}(a)} a_0\psi_0 \cdot \eta^{d_g^m(a)-1} \\
&\quad - \sum_{\overline{\Gamma}\in \mathrm{LG}_1^{\mathfrak{R}(m)}(g,a)} \frac{m(\overline{\Gamma})}{|\mathrm{Aut}(\overline{\Gamma})|} \int_{\mathbb{P}\Xi(\overline{\Gamma})^{[0]}} (\eta_{\overline{\Gamma}}^{[0]})^{d_B-1} \int_{\mathbb{P}\Xi(\overline{\Gamma})^{[-1]}} (a_0\psi_0)^0 \\
&= \int_{\mathbb{P}\Xi_g^{\mathfrak{R}(m)}(a)} (a_0\psi_0)^{d_g^m(a)} \\
&\quad - \sum_{\overline{\Gamma}\in \mathrm{LG}_1^{\mathfrak{R}(m)}(g,a)} \frac{m(\overline{\Gamma})}{|\mathrm{Aut}(\overline{\Gamma})|} \int_{\mathbb{P}\Xi(\overline{\Gamma})^{[0]}} (\eta_{\overline{\Gamma}}^{[0]})^{d_{\overline{\Gamma}}^{[0]}} \int_{\mathbb{P}\Xi(\overline{\Gamma})^{[-1]}} (a_0\psi_0)^{d_{\overline{\Gamma}}^{[-1]}},
\end{aligned} \tag{50}$$

where we used the conversion factor (16) to pass from class of boundary divisors to the integrals on their level strata and where the last equality follows by repeating the replacement procedure of η by ψ_0 a total number of $d_g^m(a)$ times. The crucial observation is that for a given graph $\overline{\Gamma} \in \mathrm{LG}_1^{\mathfrak{R}(m)}(a)$ the product of the two integrals in the sum vanishes for dimension reasons unless we are in the step number $d_{\overline{\Gamma}}^{[-1]}$ of the replacement procedure. Note that for this argument it is important that ψ_0 is always supported on bottom level.

Now the bottom levels $\mathbb{P}\Xi(\overline{\Gamma})^{[-1]}$ of every graph of a stratum with only one positive entry in its signature is again a stratum of the same type (in particular the level -1 of $\overline{\Gamma}$ consists of a single vertex), so we can re-apply the previous relation to the last term $\int_{\mathbb{P}\Xi(\overline{\Gamma})^{[-1]}}(a_0\psi_0)^{d_{\overline{\Gamma}}^{[-1]}}$. We then obtain:

$$\begin{aligned}\int_{\mathbb{P}\Xi_g^{\mathfrak{R}(m)}(a)}(a_0\psi_0)^{d_g^m(a)} &= \int_{\mathbb{P}\Xi_g^{\mathfrak{R}(m)}(a)}\eta^{d_g^m(a)} \\ &+ \sum_{\overline{\Gamma}\in \mathrm{LG}_1^{\mathfrak{R}(m)}(g,a)} \frac{m(\overline{\Gamma})}{|\mathrm{Aut}(\overline{\Gamma})|}\int_{\mathbb{P}\Xi(\overline{\Gamma})^{[0]}}(\eta_{\overline{\Gamma}}^{[0]})^{d_{\overline{\Gamma}}^{[0]}}\int_{\mathbb{P}\Xi(\overline{\Gamma})^{[-1]}}(\eta_{\overline{\Gamma}}^{[-1]})^{d_{\overline{\Gamma}}^{[-1]}} \\ &+ \sum_{\overline{\Gamma}\in \mathrm{LG}_2^{\mathfrak{R}(m)}(g,a)} \frac{m(\overline{\Gamma})}{|\mathrm{Aut}(\overline{\Gamma})|}\int_{\mathbb{P}\Xi(\overline{\Gamma})^{[0]}}(\eta_{\overline{\Gamma}}^{[0]})^{d_{\overline{\Gamma}}^{[0]}}\int_{\mathbb{P}\Xi(\overline{\Gamma})^{[-1]}}(\eta_{\overline{\Gamma}}^{[-1]})^{d_{\overline{\Gamma}}^{[-1]}}\int_{\mathbb{P}\Xi(\overline{\Gamma})^{[-2]}}(a_0\psi_0)^{d_{\overline{\Gamma}}^{[-2]}}.\end{aligned}$$

Repeating this procedure $d_g^m(a)$ times, we get the claim of the proposition. For this, a crucial observation is that for any $L \geq 0$, the level graphs obtained by gluing an arbitrary graph with 1 level passage into the bottom vertex of a graph in $\mathrm{LG}_L^{\mathfrak{R}(m)}(a)$ precisely form the set $\mathrm{LG}_{L+1}^{\mathfrak{R}(m)}(a)$ and each graph in $\mathrm{LG}_{L+1}^{\mathfrak{R}(m)}(a)$ is obtained like this in a unique way. □

We apply the previous proposition in order to simplify the expression for the Euler characteristic shown in [CMZ20a] specialized to the case of strata of abelian differentials with only one positive entry in their signature. The original formula expresses the Euler characteristic as a sum over all possible level graphs of some intersection numbers, while here we simplify the problem by expressing the Euler characteristic as a sum indexed by level graph with $L = 1$.

Corollary 6.5. *If $(a_0, a_1, \ldots, a_n)$ is a vector such that $a_0 > 0$ and $a_i \leq 0$ for all $i = 1, \ldots, n$, and $1 \leq m \leq n$, then we have:*

$$\begin{aligned}(-1)^{d_g^m(a)}\chi(\mathcal{M}_g^{\mathfrak{R}(m)}(a)) &= (d_g^m(a)+1)\int_{\mathbb{P}\Xi_g^{\mathfrak{R}(m)}(a)}(a_0\psi_0)^{d_g^m(a)} \\ &- \sum_{\overline{\Gamma}\in \mathrm{LG}_1^{\mathfrak{R}(m)}(g,a)}(d_{\overline{\Gamma}}^{[-1]}+1)\frac{m(\overline{\Gamma})}{|\mathrm{Aut}(\overline{\Gamma})|}\int_{\mathbb{P}\Xi(\overline{\Gamma})^{[0]}}(\eta_{\overline{\Gamma}^{[0]}})^{d_{\overline{\Gamma}}^{[0]}}\int_{\mathbb{P}\Xi(\overline{\Gamma})^{[-1]}}(a_0\psi_0)^{d_{\overline{\Gamma}}^{[-1]}}\end{aligned}$$

Proof. To start, let us recall the formula for the Euler characteristic of $\mathcal{M}_g^{\mathfrak{R}(m)}(a)$ proven in [CMZ20a, Theorem 1.3]:
(51)

$$(-1)^{d_g^m(a)}\chi(\mathcal{M}_g^{\mathfrak{R}(m)}(a)) = \sum_{L=0}^{d_B}\sum_{\overline{\Gamma}'\in \mathrm{LG}_L^{\mathfrak{R}(m)}(g,a)}(d_{\overline{\Gamma}'}^{[0]}+1)\frac{m(\overline{\Gamma}')}{|\mathrm{Aut}(\overline{\Gamma}')|}\prod_{i=0}^{-L}\int_{\mathbb{P}\Xi(\overline{\Gamma}')^{[i]}}(\eta_{\overline{\Gamma}'}^{[i]})^{d_{\overline{\Gamma}'}^{[i]}}$$

Given $\overline{\Gamma}' \in \mathrm{LG}_L^{\mathfrak{R}(m)}(g,a)$ for $L \geq 1$ as in the formula above, we write $\delta_1(\overline{\Gamma}') = \overline{\Gamma}$ for the first undegeneration of $\overline{\Gamma}'$, that is the 1-level graph $\overline{\Gamma}$ obtained from $\overline{\Gamma}'$ by contracting all level passages of $\overline{\Gamma}'$ apart from the first one. To prove the formula from the corollary, we group all summands of (51) associated to non-trivial graphs $\overline{\Gamma}'$ according to their undegeneration $\overline{\Gamma} = \delta_1(\overline{\Gamma}')$.

To simplify the formula from there, observe that given any $\overline{\Gamma} \in \mathrm{LG}_1^{\mathfrak{R}(m)}(g,a)$, we obtain the equality

$$\frac{m(\overline{\Gamma})}{|\mathrm{Aut}(\overline{\Gamma})|} \cdot \int_{\mathbb{P}\Xi(\overline{\Gamma})^{[0]}} (\eta_{\overline{\Gamma}^{[0]}})^{d_{\Gamma}^{[0]}} \cdot \int_{\mathbb{P}\Xi(\overline{\Gamma})^{[-1]}} (a_0\psi_0)^{d_{\overline{\Gamma}}^{[-1]}}$$
$$= \sum_{L=1}^{d_g^m(a)} \sum_{\substack{\overline{\Gamma}' \in \mathrm{LG}_L^{\mathfrak{R}(m)}(g,a) \\ \delta_1(\overline{\Gamma}')=\overline{\Gamma}}} \frac{m(\overline{\Gamma}')}{|\mathrm{Aut}(\overline{\Gamma}')|} \prod_{i=0}^{-L} \int_{\mathbb{P}\Xi(\overline{\Gamma}')^{[i]}} (\eta_{\overline{\Gamma}'}^{[i]})^{d_{\overline{\Gamma}'}^{[i]}}.$$

by applying Proposition 6.4 to the bottom level stratum $\mathbb{P}\Xi(\overline{\Gamma})^{[-1]}$. Multiplying this equality by $(d_{\overline{\Gamma}}^{[0]}+1)$ and summing over all choices of $\overline{\Gamma}$, we obtain all summands of the right-hand side of (51) associated to non-trivial graphs $\overline{\Gamma}'$. In particular, writing the term of the trivial graph separately, we obtain

$$(-1)^{d_g^m(a)}\chi(\mathcal{M}_g^{\mathfrak{R}(m)}(a)) = (d_g^m(a)+1)\int_{\mathbb{P}\Xi_g^{\mathfrak{R}(m)}(a)} \eta^{d_g^m(a)} +$$
$$+ \sum_{\overline{\Gamma}\in \mathrm{LG}_1(g,a)} (d_{\overline{\Gamma}}^{[0]}+1)\frac{m(\overline{\Gamma})}{|\mathrm{Aut}(\overline{\Gamma})|}\int_{\mathbb{P}\Xi(\overline{\Gamma})^{[0]}} (\eta_{\overline{\Gamma}}^{[0]})^{d_{\overline{\Gamma}}^{[0]}} \int_{\mathbb{P}\Xi(\overline{\Gamma})^{[-1]}} (a_0\psi_0)^{d_{\overline{\Gamma}}^{[-1]}}$$

In order to show the main statement, it is enough to rewrite $\int_{\mathbb{P}\Xi_g^{\mathfrak{R}(m)}(a)} \eta^{d_g^m(a)}$ using the relation (50) and using that $d_g^m(a) - d_{\overline{\Gamma}}^{[0]} = d_{\overline{\Gamma}}^{[-1]} + 1$. □

Remark 6.6. Following a similar approach, one can show that the Euler characteristic of a general stratum (not necessarily with only one zero), may be written as a sum only on two level graphs (even more, only the ones with only one vertex on top level). However this sum would contain integrals of powers of η on meromorphic strata which are a priori hard to compute.

Now, we apply Corollary 6.5 to the case of the minimal stratum $\mathcal{M}_g(2g-1)$. In this case, the top level of a 2-level graph is basically given by a product of holomorphic strata. Then, as we explain below, we can use the fact that top-η powers of non-minimal holomorphic strata vanish thanks to [Sau18], to simplify further the expression of the Euler characteristic.

Proposition 6.7. *For $g \geq 1$, we denote $d_g = 2g-1$. Then for all $g \geq 1$, we have:*

$$\chi(\mathcal{M}_g(d_g)) = 2g \cdot d_g^{d_g-1} \cdot \mathcal{A}_g(d_g)$$
$$+ \sum_{g^\perp=1}^{g-1} 2g^\perp \sum_{\substack{\underline{g}=(g_1,\dots,g_n) \\ |\underline{g}|=g-g^\perp}} \frac{(-1)^{n+1}}{n!} d_g^{d_{g^\perp}} \mathcal{A}_{g^\perp}^{\mathfrak{R}(n-1)}(d_g,-d_{\underline{g}}) \prod_{i=1}^n d_{g_i} \int_{\mathbb{P}\Xi_{g_i}(d_{g_i})} \eta^{d_{g_i}}$$

where the second sum is on all vectors of size size $g-g^\perp$, and $d_{\underline{g}} = (d_{g_1},\dots,d_{g_n})$.

Proof. We want to describe the 2-level graphs which give a non-trivial contribution in Corollary 6.5. In order to do that, we want to investigate top η-powers on top level strata of 2-level graphs.

Since these level strata are in general disconnected, we first reduce the computation of powers of η on disconnected strata in terms of powers of η on connected ones. For a graph $\overline{\Gamma} \in \mathrm{LG}_1(g,(d_g))$, the total space of the top level stratum $\mathbb{P}\Xi(\overline{\Gamma})^{[0]}$ is isomorphic to the total space of the projective bundle $\pi : \mathbb{P}(\mathcal{E}_{\overline{\Gamma}}) \to Y_{\overline{\Gamma}}$ where

$$Y_{\overline{\Gamma}} = \prod_{v \in V(\overline{\Gamma}):\ l(v)=0} \mathbb{P}\Xi_{g(v)}(I(v))$$

and

$$\mathcal{E}_{\overline{\Gamma}} = \bigoplus_{v \in V(\overline{\Gamma}):\ l(v)=0} p_v^* \mathcal{O}_{\mathbb{P}\Xi_{g(v)}(I(v))}(-1),$$

with $p_v : Y_{\overline{\Gamma}} \to \mathbb{P}\Xi_{g(v)}(I(v))$ being the projection map. Then, we have:

$$\begin{aligned} \int_{\mathbb{P}\Xi(\overline{\Gamma})^{[0]}} \eta^{d_{\overline{\Gamma}}^{[0]}} &= (-1)^{d_{\overline{\Gamma}}^{[0]}} \int_{Y_{\overline{\Gamma}}} s_{d_{\overline{\Gamma}}^{[0]} - v_{\overline{\Gamma}}^{[0]} + 1}(\mathcal{E}_{\overline{\Gamma}}) \\ &= (-1)^{v_{\overline{\Gamma}}^{[0]}+1} \prod_{v \in V(\overline{\Gamma}):\ l(v)=0} \int_{\mathbb{P}\Xi_{g(v)}(I(v))} \eta^{d_{g(v)} + |I(v)| - 1}, \end{aligned} \tag{52}$$

where $v_{\overline{\Gamma}}^{[0]}$ is the number of vertices of level 0.

By [Sau18], the top η-powers on non-minimal holomorphic strata vanish. Hence, thanks to (52), the only non trivial contribution from a 2-level graph on the right hand side of the statement of Corollary 6.5 is given by star-shaped graphs, i.e. 2-level graphs with only one vertex on bottom level (this is the only possiblity in a stratum with only one positive entry in its signature) and with some vertices on top level, each top vertex connected to the bottom vertex via only one edge.

We can parametrize such 2-level graphs by choosing a partition $(g_1, \ldots, g_n)$ of the possible total genus $g^\top$ on top level.[9] Note that in the formula the number $g^\top$ was rewritten as the difference $g - g^\perp$. The bottom level stratum of a 2-level graph defined by a partition $(g_1, \ldots, g_n)$ is given by $\mathbb{P}\Xi^{\mathfrak{R}(n-1)}_{g^\perp, n+1}(2g-1, -2g_1 + 1, \ldots, -2g_n + 1)$, since the global residue condition imposes zero residue as every pole. We conclude by using (52) and rewriting the top ψ_0 contributions using the symbols previously introduced. Note that in the contribution from the trivial graph, the term $\int_{\overline{\mathcal{M}}_g(d_g)} \psi_1^{d_g}$ has to be replaced by $-\mathcal{A}_g(d_g)/d_g$, since $\mathcal{M}_g(2g-1)$ is a holomorphic stratum (see the relation (4)). □

We may now state and prove the following theorem, which is equivalent to Theorem 1.5 stated in the introduction after using coefficient extraction formulas.

Theorem 6.8. *For all $g \geq 1$, we have*

$$\chi(\mathcal{M}_g(d_g)) = \sum_{\substack{1 \leq g^\perp \leq g \\ n \geq 0}} 2g^\perp \sum_{\substack{\underline{g} = (g_1, \ldots, g_n) \\ \sum g_i = g - g^\perp}} \frac{(-1)^n}{n!} d_g^{d_{g^\perp} + n - 1} \mathcal{A}_{g^\perp}(d_g, -d_{\underline{g}}) \prod_{i=1}^n (d_{g_i})! \cdot b_{g_i}$$

where $d_{\underline{g}} = (d_{g_1}, \ldots, d_{g_n})$ and where the numbers b_{g_i} are determined by

$$\frac{z/2}{\sinh(z/2)} = 1 + \sum_{g \geq 0} b_g z^{2g}\,.$$

[9] We remark that a 2-level graph only determines an unordered partition $g^\top = g_1 + \ldots + g_n$. By a standard combinatorial argument using the Orbit-Stabilizer theorem, summing over such partitions and weighting by $1/|\mathrm{Aut}(\overline{\Gamma})|$ as in Corollary 6.5 is equivalent to summing over vectors $(g_1, \ldots, g_n)$ and dividing by $n!$ as in Proposition 6.7.

Before giving the proof of this theorem we introduce a set of twisted graphs that will be used only in this proof and in the proof of Lemma 6.9 below. We fix $g \geq 1$ and $0 \leq g^{\perp} \leq g$. The set $\widetilde{\mathrm{TR}}(g, g^{\perp})$ is the set of graphs of compact type (trees) of genus g with 1 leg satisfying the following properties:

- the root $v^{\perp}$ of the tree has genus $g^{\perp}$ and carries the unique leg;
- the leaves of the tree have positive genus (we denote by $\mathrm{Leaf}(\Gamma)$ the set of leaves);
- internal nodes (vertices which are neither the root nor leaves) have genus 0 (this set will be denoted by $\mathrm{Int}(\Gamma)$).

Each such graph carries a unique twist such that the twist at the unique leg is $2g-1$. Moreover, we define $\widetilde{\mathrm{TR}}(g)$ to be the set obtained from $\widetilde{\mathrm{TR}}(g, 0)$ by adding the trivial graph in genus g with one leg.

Proof. We start from Proposition 6.7. First we will use Proposition 6.2 to replace integrals on strata with residue conditions by integrals on classical strata.

$$\mathcal{A}_g^{\mathfrak{R}(n-1)}(d_g, -d_{\underline{g}}) = -\sum_{(\Gamma, I) \in \mathrm{TR}(g^{\perp}, (d_g, -d_{\underline{g}}))} \mathcal{A}_g(I(v_0)) \cdot F(\Gamma, I). \tag{53}$$

Note that if we define $\mathcal{A}_g^{\mathfrak{R}(-1)}(d_g) = \mathcal{A}_g(d_g)/d_g$, then it is immediate from the definition of $F(\Gamma, I)$ that equation (53) is still satisfied in this case. Moreover the first term in Proposition 6.7 can be written as

$$2g \cdot d_g^{d_g - 1} \cdot \mathcal{A}_g(d_g) = 2g \cdot d_g^{d_g} \cdot \mathcal{A}_g^{\mathfrak{R}(-1)}(d_g)$$

and can then be integrated as the $g^{\perp} = g, n = 0$ term of the remaining sum.

By inserting equality (53) into the formula of Proposition 6.7, we can replace the expression of $\chi(\mathcal{M}_g(d_g))$ as a sum over star-shaped graphs where one central vertex contributes with a function $\mathcal{A}$ with residue condition, by a sum indexed by the sets $\widetilde{\mathrm{TR}}(g, g^{\perp})$ where the central vertex contributes with the function $\mathcal{A}$ (without residues):

(54)

$$\chi(\mathcal{M}_g(d_g)) = \sum_{1 \leq g^{\perp} \leq g} 2g^{\perp} \sum_{\Gamma \in \widetilde{\mathrm{TR}}(g, g^{\perp})} \frac{(-1)^{n(v^{\perp})+1}}{|\mathrm{Aut}(\Gamma)|} d_g^{d_{g^{\perp}} + n(v^{\perp}) - 2} \mathcal{A}_{g^{\perp}}(I(v^{\perp}))$$
$$\times \left(\prod_{v \in \mathrm{Int}(\Gamma)} -(-I(v)^{+})^{n(v)-2} \right) \times \left(\prod_{v \in \mathrm{Leaf}(\Gamma)} d_{g(v)} a_{g(v)} \right),$$

where, as above we used the notation $I(v)^{+}$ for the unique positive twist at an internal vertex, and we used the notation:

$$a_g = -\int_{\mathbb{P}\Xi_g(2g-1)} \eta^{2g-1},$$

for all $g \geq 1$.

We will regroup this sum as follows. In order to construct a tree in $\widetilde{\mathrm{TR}}(g, g^{\perp})$, one may equivalently chose: (i) the number $n = n(v^{\perp}) - 1 \geq 0$ of descendants of the root; (ii) the genera $g_1, g_2, \ldots, g_n$ of the descendants (i.e. the n trees obtained by removing the root); (iii) a tree in $\widetilde{\mathrm{TR}}(g_i)$ for all $1 \leq i \leq n$. An autormorphism of this new data is given by a permutation σ of $\{1,\dots,n\}$ and an isomorphism of Γ_i and $\Gamma_{\sigma(i)}$ for all $1\leq i\leq n$.

Then formula (54) becomes:

$$\chi(\mathcal{M}_g(d_g)) = \sum_{\substack{1\leq g^\perp\leq g\\ n\geq 0}} 2g^\perp \sum_{\substack{\underline{g}=(g_1,\dots,g_n)\\ \sum g_i=g-g^\perp}} \frac{(-1)^n}{n!} d_g^{d_{g^\perp}+n-1}\mathcal{A}_{g^\perp}(d_g,-d_{\underline{g}})$$
$$\times \prod_{i=1}^n \sum_{\Gamma_i\in\widetilde{\mathrm{TR}}(g_i)} \frac{1}{|\mathrm{Aut}(\Gamma_i)|}\left(\prod_{v\notin \mathrm{Leaf}(\Gamma_i)} -(-I(v)^+)^{n(v)-2}\right)\times\left(\prod_{v\in\mathrm{Leaf}(\Gamma_i)} d_{g(v)}a_{g(v)}\right).$$

Thus, the theorem follows from Lemma 6.9 below. □

Lemma 6.9. *For all $g\geq 1$, we have:*

$$(2g-1)!b_g = \sum_{\Gamma\in\widetilde{\mathrm{TR}}(g)} \frac{1}{|\mathrm{Aut}(\Gamma)|}\left(\prod_{v\notin \mathrm{Leaf}(\Gamma)} -(-I(v)^+)^{n(v)-2}\right)\times\left(\prod_{v\in\mathrm{Leaf}(\Gamma)} d_{g(v)}a_{g(v)}\right).$$

Proof. We first recall from [Sau18, Theorem 1.6] that the series b_g and a_g satisfy the following identity for all $g\geq 1$

$$[z^{2g}]\mathcal{F}(z)^{2g} = (2g)!b_g, \tag{55}$$

where $\mathcal{F}(z) = 1+\sum_{g\geq 1}(2g-1)a_g z^{2g}$.[10]

We will prove the lemma by induction on $g\geq 1$. First, we note that the initial step is trivial as $b_1 = a_1$ and there is only one (trivial) graph in $\widetilde{\mathrm{TR}}(1)$.

Let $g\geq 2$. As in the previous proof, we may decompose a sum over non-trivial graphs in $\widetilde{\mathrm{TR}}(g)$ as a sum over: (i) choice of $n\geq 2$; (ii) a vector $(g_1,\dots,g_n)$ of size g; (iii) choices of n trees in $\widetilde{\mathrm{TR}}(g_i)$. Thus we get:

$$\begin{aligned}
&\sum_{\Gamma\in\widetilde{\mathrm{TR}}(g)} \frac{1}{|\mathrm{Aut}(\Gamma)|}\left(\prod_{v\notin \mathrm{Leaf}(\Gamma)} -(-I(v)^+)^{n(v)-2}\right)\times\left(\prod_{v\in\mathrm{Leaf}(\Gamma)} d_{g(v)}a_{g(v)}\right)\\
=\ & d_g a_g - \sum_{\substack{n\geq 2\\ g_1+\dots+g_n=g}} \frac{(-2g+1)^{n-1}}{n!}\prod_{i=1}^n \sum_{\Gamma_i\in\widetilde{\mathrm{TR}}(g_i)}\frac{1}{|\mathrm{Aut}(\Gamma_i)|}\\
&\left(\prod_{v\notin \mathrm{Leaf}(\Gamma_i)} -(-I(v)^+)^{n(v)-2}\right)\times\left(\prod_{v\in\mathrm{Leaf}(\Gamma_i)} d_{g(v)}a_{g(v)}\right)\\
=\ & d_g a_g - \sum_{\substack{n\geq 2\\ g_1+\dots+g_n=g}} \frac{(-2g+1)^{n-1}}{n!}\prod_{i=1}^n (2g_i-1)!b_{g_i}.
\end{aligned}$$

The last line was obtained by applying the induction hypothesis. Thus we need to show that

$$(2g-1)a_g = \sum_{\substack{n\geq 1\\ g_1+\dots+g_n=g}} \frac{(-2g+1)^{n-1}}{n!}\prod_{i=1}^n (2g_i-1)!b_{g_i} \tag{56}$$

[10] Note that the paper [Sau18] works with the class $\xi=-\eta$ in the definition of its numbers a_g and the function $(t/2)/\sin(t/2)$ for its analogue of the numbers b_g, both resulting in a factor $(-1)^g$ compared to our definition.

for all $g \geq 2$. This last identity is equivalent to formula (55) by [BGW19, Corollary 2.(ii)], which we apply to $x_n = (2n-1)a_n, y_n = (2n-1)!b_n$, and $(a,b,c,d) = (-2,0,0,1)$ in the notation of this paper, thus finishing the proof. □

Finally, we are able to prove the version of the formula for the Euler characteristic that is claimed in the introduction.

Proof of Theorem 1.5. We use the expression of $\chi(\mathcal{M}_g(2g-1))$ given by Theorem 6.8 to obtain the coefficient extraction formula of Theorem 1.5. We begin with a slight rearrangement of terms:

$$\begin{aligned} &\sum_{\substack{1\leq g^\perp \leq g\\ n\geq 0}} 2g^\perp \sum_{\substack{\underline{g}=(g_1,\dots,g_n)\\ \sum g_i = g-g^\perp}} \frac{(-1)^n}{n!} d_g^{d_{g^\perp}+n-1} \mathcal{A}_{g^\perp}(d_g, -d_{\underline{g}}) \prod_{i=1}^n (d_{g_i})! \cdot b_{g_i} \\ = \; & d_g^{2g-2} \sum_{\substack{1\leq g^\perp \leq g\\ n\geq 0}} \frac{2g^\perp}{n!} \sum_{\substack{\underline{g}=(g_1,\dots,g_n)\\ \sum g_i = g-g^\perp}} \mathcal{A}_{g^\perp}(d_g, -d_{\underline{g}}) \prod_{i=1}^n -d_g^{1-2g_i}(d_{g_i})! \cdot b_{g_i}. \end{aligned}$$

Now, we use Theorem 1.1 to write the function $\mathcal{A}_{g^\perp}$ with a coefficient extraction formula. Then the $\chi(\mathcal{M}_g(2g-1))$ takes the following shape:

$$\begin{aligned} & d_g^{2g-2} \sum_{\substack{1\leq g^\perp \leq g\\ n\geq 0}} \frac{2g^\perp}{n!} [z^{2g^\perp}] \frac{\exp(d_g z\mathcal{S}'(z)/\mathcal{S}(z))}{\mathcal{S}(z)^{2g+1}} \\ & \sum_{\substack{\underline{g}=(g_1,\dots,g_n)\\ \sum g_i = g-g^\perp}} \prod_{i=1}^n -\mathcal{S}(d_{g_i}z)(d_g/\mathcal{S}(z))^{1-2g_i}(d_{g_i})! \cdot b_{g_i} \\ = \; & d_g^{2g-2}[z^{2g}] \frac{\exp(d_g z\mathcal{S}'(z)/\mathcal{S}(z))}{\mathcal{S}(z)^{2g+1}} \\ & \sum_{1\leq g^\perp\leq g} 2g^\perp [y^{-2(g-g^\perp)}] \exp\left(\sum_{\widetilde{g}>0} -\left(\frac{z}{y}\right)^{2\widetilde{g}} \mathcal{S}(d_{\widetilde{g}}z)(d_g/\mathcal{S}(z))^{1-2\widetilde{g}}(d_{\widetilde{g}})! \cdot b_{\widetilde{g}}\right) \\ = \; & d_g^{2g-2}[z^{2g}] \frac{\exp(d_g z\mathcal{S}'(z)/\mathcal{S}(z))}{\mathcal{S}(z)^{2g+1}} \\ & \sum_{1\leq g^\perp\leq g} 2g^\perp [y^{-2(g-g^\perp)}] \exp\left(-d_g\mathcal{H}(d_g y, z)\right) \\ = \; & d_g^{2g-2}[z^{2g}] \frac{\exp(d_g z\mathcal{S}'(z)/\mathcal{S}(z))}{\mathcal{S}(z)^{2g+1}} \left(\left(y+1+y\frac{\partial}{\partial y}\right)\exp\left(-d_g\mathcal{H}\right)\right)\Big|_{y=2g-1} \\ = \; & d_g^{2g-2}[z^{2g}] \left(\frac{y+1-y^2\frac{\partial \mathcal{H}}{\partial y}}{\mathcal{S}(z)^2} \exp\left(y\left(\frac{z\mathcal{S}'(z)}{\mathcal{S}(z)} - \ln(\mathcal{S}(z)) - \mathcal{H}\right)\right)\right)\Big|_{y=2g-1}. \end{aligned}$$

Note that in the third equality we use the fact that for a Laurent series $G(y,z)$ such that all Laurent monomials $y^e z^f$ satisfy $f \geq -e > 0$, which we later specialize to $G(y,z) = \exp(-d_g\mathcal{H}(y,z))$, we have that

$$[z^{2g}] \sum_{1\leq g^\perp\leq g} 2g^\perp [y^{-2(g-g^\perp)}] G(d_g y, z) = [z^{2g}](y+1+y\frac{\partial}{\partial y})G(y,z)\Big|_{y=2g-1}. \tag{57}$$

The equality (57) can be checked on Laurent monomials. □

6.3. **Euler characteristic of connected components of minimal strata.** We want to refine the previous Euler characteristic computation and compute the Euler characteristic of the connected components of minimal strata. In order to do so, we show first a spin version of Theorem 6.8. Recall that we set

$$\chi(\mathcal{M}_g(2g-1))^{\mathrm{spin}} := \chi(\mathcal{M}_g(2g-1)^{\mathrm{even}}) - \chi(\mathcal{M}_g(2g-1)^{\mathrm{odd}})$$

to be difference of the orbifold Euler characteristics of the even and odd components of the minimal stratum $\mathcal{M}_g(2g-1)$.

Theorem 6.10. *Assume part (4) of Assumption 1.3 holds, then for all $g \geq 1$, we have*

$$\chi(\mathcal{M}_g(d_g))^{\mathrm{spin}} = \sum_{\substack{1\leq g^\perp \leq g\\ n\geq 0}} 2g^\perp \sum_{\substack{\underline{g}=(g_1,\dots,g_n)\\ \sum g_i = g-g^\perp}} \frac{(-1)^n}{n!} d_g^{d_{g^\perp}+n-1} \mathcal{A}^{\mathrm{spin}}_{g^\perp}(d_g, -d_{\underline{g}}) \prod_{i=1}^n (d_{g_i})! \cdot \widetilde{b}_{g_i}$$

where $d_g = 2g-1$, $\widetilde{b}_g = \frac{-2^{g-1}}{2^{2g-1}-1} b_g$ and $d_{\underline{g}} = (d_{g_1}, \dots, d_{g_n})$.

Proof. First of all recall that by a generalized version of the Gauss-Bonnet theorem (see for example [CMZ20a, Sec. 2]), we have

$$(-1)^{d_g(a)} \chi(\mathcal{M}_g^{\mathrm{spin}}(a)) = c_{d_g(a)}(\Omega^1_{\mathbb{P}\Xi_g(a)}(\log(D))) \cdot [\mathbb{P}\Xi_g^{\mathrm{spin}}(a)]$$

where $D = \mathbb{P}\Xi_g(a) \setminus \mathcal{M}_g(a)$. The top Chern class of the logarithmic cotangent bundle $\Omega^1_{\mathbb{P}\Xi_g(a)}(\log(D))$ was computed (as a cohomology class) in [CMZ20a, Theorem 9.10]. It can be written as a sum over all level graphs $\overline{\Gamma}'$ for $\mathbb{P}\Xi_g(a)$ as in the right-hand side of (51), with integrals replaced by gluing pushforwards. As in the proof of Corollary 6.5 we can then regroup the summands according to their first undegeneration $\overline{\Gamma}$ and use Proposition 6.4 to obtain

$$c_{d_g(a)}(\Omega^1_{\mathbb{P}\Xi_g(a)}(\log(D))) = (d_g(a)+1) \cdot (a_0\psi_0)^{d_g(a)} \cdot [\mathbb{P}\Xi_g(a)] - \sum_{\overline{\Gamma}\in \mathrm{LG}_1(g,a)} (d_{\overline{\Gamma}}^{[-1]}+1) \cdot \ell_{\overline{\Gamma}} \cdot \eta^{d_{\overline{\Gamma}}^{[0]}} \cdot (a_0\psi_0)^{d_{\overline{\Gamma}}^{[-1]}} \cdot [\mathbb{P}\Xi(\overline{\Gamma})] \tag{58}$$

Here we note that while Proposition 6.4 was formulated on the level of intersection numbers, the proof only uses relations in the Chow groups and can thus be lifted to an equality of cohomology classes.

Now we want to argue that, as in the proof of Proposition 6.7, all non-star graphs in the sum above have a trivial contribution. To see this, first note that when pairing the class (58) with $[\overline{\mathcal{M}}_g(a)]^{\mathrm{spin}}$, the fundamental class of each connected component of the loci $\mathbb{P}\Xi(\overline{\Gamma})$ is simply multiplied by ± 1, depending on the parity of the associated differential. Then, we may carry out the argument in Proposition 6.7, which converts the top power of η on $\mathbb{P}\Xi_{g(v)}(I(v))^{[0]}$ into a product over the level zero strata, on each component of $\mathbb{P}\Xi_{g(v)}(I(v))^{[0]}$ separately. Finally, we observe that the proofs explained in Section 3.1 of [Sau18] show that the top power of η vanishes *on each connected component* of every non-minimal stratum of holomorphic differentials. Thus, even in the spin setting, only terms coming from star graphs $\overline{\Gamma}$ can give non-trivial contributions in (58).

Since these terms are all of compact type, we know that the parity under the associated gluing map is simply the sum of parities at all vertices by Proposition

5.2. This implies that we can now repeat the proof of Proposition 6.7, replacing all fundamental classes by their spin counterparts, and arrive at the following formula:

$$\chi(\mathcal{M}_g^{\mathrm{spin}}(d_g)) = 2g \cdot d_g^{d_g-1} \cdot \mathcal{A}_g^{\mathrm{spin}}(d_g)$$
$$+ \sum_{g^\perp=1}^{g-1} 2g^\perp \sum_{\substack{\underline{g}=(g_1,\dots,g_n) \\ |\underline{g}|=g-g^\perp}} \frac{(-1)^{n+1}}{n!} d_g^{d_{g^\perp}} \mathcal{A}_{g^\perp}^{\mathrm{spin},\mathfrak{R}(n-1)}(d_g, -d_{\underline{g}}) \prod_{i=1}^n d_{g_i} \int_{\mathbb{P}\Xi_{g_i}^{\mathrm{spin}}(d_{g_i})} \eta^{d_{g_i}}.$$

Note that in order to obtain the first term above, we use part (4) of Assumption 1.3 to replace $\int_{[\overline{\mathcal{M}}_g(d_g)]^{\mathrm{spin}}} \psi_1^{d_g}$ by $-\mathcal{A}_g^{\mathrm{spin}}(d_g)/d_g$.

Before continuing, we remark that Lemma 6.1 and its consequence Proposition 6.2 are true in the spin case after substituting the terms with their spin counterparts. Indeed the only graphs appearing are of compact type, and so again the parity of the graph is given by the sum of the parities of its vertices. Moreover, when generalizing the proof of Proposition 6.2 we use in the end that in genus zero, the spin cycle agrees with the fundamental class $[\overline{\mathcal{M}}_{0,n}]$.

Using the analogue of Proposition 6.2, we can now run the same argument as in the proof of Theorem 6.8 to reduce to the spin-version of Lemma 6.9, where b_g is replaced by $\widetilde{b}_g$ and a_g is replaced by

$$a_g^{\mathrm{spin}} = -\int_{[\mathbb{P}\Xi(2g-1)]^{\mathrm{spin}}} \eta^{2g-1}.$$

In the proof of this lemma, the role of equality (56) is played by the corresponding equality

$$(2g-1)a_g^{\mathrm{spin}} = \sum_{\substack{n\geq 1 \\ g_1+\dots+g_n=g}} \frac{(-2g+1)^{n-1}}{n!} \prod_{i=1}^n (2g_i-1)!\widetilde{b}_{g_i}$$

which is proved in [CMSZ20, Corollary 6.11] by means of representation theory. Indeed, from [CMZ20a] and [Sau18], the intersection number a_g^{spin} is (up to a simple combinatorial coefficient) the difference between the Masur-Veech volumes of the odd and even component of $\mathcal{M}_g(2g-1)$ computed in [CMSZ20].

The final observation that we need is that when generalizing the proof of Lemma 6.9, which was an induction in $g \geq 1$, the initial equation to check here is $\widetilde{b}_1 = a_1^{\mathrm{spin}}$. On the one hand, from the formula we have $\widetilde{b}_1 = -b_1$. On the other hand, the equality $[\overline{\mathcal{M}}_1(1)]^{\mathrm{spin}} = -[\overline{\mathcal{M}}_{1,1}]$ following from (13) implies $a_1^{\mathrm{spin}} = -a_1$ and so we can conclude from the known equality $b_1 = a_1$. ☐

Using this theorem, the proof of Theorem 1.6 is analogous to the proof of Theorem 1.5 given in the previous section. Here we use Theorem 1.4 to explicitly compute the function $\mathcal{A}_{g^\perp}^{\mathrm{spin}}$ and thus we need parts (1) to (3) of Assumption 1.3.

Note that one can then compute the Euler characteristic of all the connected components of minimal strata. Indeed recall from [KZ03, Cor. 1] that for $g = 2$ there is only the hyperelliptic component, for $g = 3$ there is hyperelliptic and the odd component, and for $g \geq 4$ there are exactly three components given by the hyperelliptic component and the non-hyperelliptic odd and even components. Moreover by [KZ03, Cor. 3] we know that the hyperelliptic component has even parity for odd genera and odd parity for even genera. Hence, using the previous information together with the result $\chi(\mathcal{M}_g^{\text{hyp}}(d_g)) = \frac{-1}{4g(2g+1)}$ (see [CMZ20a, Prop. 10.4]), we can use the formulas we have shown for $\chi(\mathcal{M}_g(d_g))$ and $\chi(\mathcal{M}_g(2g-1))^{\text{spin}}$ to compute the Euler characteristic of every connected component of the minimal strata.

Using the statements of the previous paragraph, one can independently compute the Euler characteristic of the spin components in genus $g = 2, 3$. One can then double-check that the values given in Table 1 using the formula of Theorem 1.6 are indeed correct in genus 2 and 3.

Appendix A. Polynomiality properties in the splitting formula

The goal of this section is to prove the following result used in the proof of Proposition 3.1.

Lemma A.1. *For $g, n \geq 0$ with $2g - 2 + n > 0$, the right-hand side of equation* (17) *from Proposition 3.1 is given by a (cycle-valued) polynomial in a.*

Before we begin, we need two technical preliminaries about sums of polynomials over partitions of given numbers. Here we remark that for the entire section we have the convention that when iterating over sums $b_1 + \ldots + b_e = c$ for e, c fixed, the terms are ordered (so that for $e = 2, c = 3$ the sums $1 + 2 = 2 + 1 = 3$ are counted separately).

Lemma A.2. *Let $f \geq 1$, then the function*

$$S_f : \mathbb{Z}_{\geq 0} \to \mathbb{Z}, c \mapsto \sum_{\substack{b_1+b_2=c \\ b_i \in \mathbb{Z}_{\geq 1}}} (b_1 b_2)^f$$

is given by a polynomial in c satisfying $S_f(-c) = -S_f(c)$.

Proof. Let $c \geq 1$ be an integer, then we write $b_1 = b, b_2 = c - b$ and expand the binomial to compute

$$\begin{aligned} S_f(c) &= \sum_{b=1}^{c-1} b^f (c-b)^f = \sum_{b=1}^{c-1} b^f \sum_{i=0}^{f} \binom{f}{i} c^{f-i} (-b)^i \\ &= \sum_{i=0}^{f} (-1)^i \binom{f}{i} c^{f-i} \sum_{b=1}^{c-1} b^{f+i} . \end{aligned}$$

Using Faulhaber's formula, we can compute the sum of the b^{f+i} in terms of Bernoulli numbers B_j, obtaining

$$\begin{aligned} S_f(c) &= \sum_{i=0}^{f} (-1)^i \binom{f}{i} c^{f-i} \frac{1}{f+i+1} \sum_{j=0}^{f+i} \binom{f+i+1}{j} B_j c^{f+i+1-j} \\ &= \sum_{i=0}^{f} \sum_{j=0}^{f+i} (-1)^i \binom{f}{i} \binom{f+i+1}{j} \frac{1}{f+i+1} B_j c^{2f+1-j} . \end{aligned}$$

Since all odd Bernoulli numbers except for B_1 vanish, the lemma is proved once we show that the coefficient of c^{2f+1-1} vanishes, since then only odd powers of c

appear above. This corresponds to extracting the terms for $j = 1$ and so indeed we obtain

$$\begin{aligned}[c^{2f}]S_f(c) &= B_1 \sum_{i=0}^{f} (-1)^i \binom{f}{i} \binom{f+i+1}{1} \frac{1}{f+i+1} \\ &= B_1 \sum_{i=0}^{f} (-1)^i \binom{f}{i} = 0\,.\end{aligned}$$

□

Lemma A.3. *Let $Q(A,B) = Q(a_1,\dots,a_n,b_1,\dots,b_e)$ be a polynomial with rational coefficients in $n+e$ variables (for $n \geq 0, e \geq 1$) satisfying $Q(-A,-B) = Q(A,B)$. Let $c : \mathbb{Q}^n \to \mathbb{Q}$ be a nonzero $\mathbb{Q}$-linear map and denote by $\Lambda_c^+ \subseteq \mathbb{Z}^n$ the set of integer vectors A for which $c(A)$ is integral and non-negative. Then the expression*

$$P(A) = \sum_{\substack{b_1+\ldots+b_e=c(A)\\ b_i \in \mathbb{Z}_{\geq 1}}} b_1 \cdots b_e \cdot Q(A,B), \quad A \in \Lambda_c^+ \tag{59}$$

is given by a polynomial expression in the entries of A, and this polynomial is divisible by $c(A)$ and satisfies $P(-A) = -P(A)$.

Proof. We prove this result by induction on e, treating the cases $e = 1, 2$ separately. For $e = 1$ the expression (59) takes the simple shape $P(A) = c(A) \cdot Q(A, c(A))$, which is clearly polynomial in A, divisible by $c(A)$ and satisfies

$$P(-A) = c(-A) \cdot Q(-A, c(-A)) = -c(A) \cdot Q(A, c(A)) = -P(A)\,,$$

using that c is linear and that Q is even.

For the case $e = 2$ we first note that we can assume without loss of generality, that Q is symmetric in the variables b_1, b_2. Indeed, since the tuples (b_1, b_2) in the sum over $b_1+b_2 = c(A)$ are symmetric under exchanging b_1, b_2, we can replace Q by the symmetric average $\tilde{Q}(A,B) = (Q(A,b_1,b_2) + Q(A,b_2,b_1))/2$ without changing the value of P.

Since the symmetric functions in b_1, b_2 are generated as an algebra by the elementary symmetric functions $b_1 + b_2$ and $b_1 b_2$, it suffices to prove the lemma for Q of the form

$$Q(A, b_1, b_2) = q(A) \cdot (b_1 + b_1)^{f_1} (b_1 b_2)^{f_2}\,,$$

where q is a polynomial, which must be even for f_1 even and odd for f_1 odd. Plugging this form into the expression for P we have

$$\begin{aligned} P(A) &= q(A) \sum_{b_1+b_2=c(A)} b_1 b_2 (b_1 + b_1)^{f_1} (b_1 b_2)^{f_2} \\ &= q(A) c(A)^{f_1} \sum_{b_1+b_2=c(A)} (b_1 b_2)^{f_2+1} = q(A) c(A)^{f_1} S_{f_2+1}(c(A))\,, \end{aligned} \tag{60}$$

where for the last equality we use the result and notation of Lemma A.2. From the form (60) we see that P is polynomial. Moreover, for f_1 either even or odd, we see that the expression $q(A)c(A)^{f_1}$ is even in A. Thus, since S_{f_2+1} is odd by Lemma A.2, the overall expression for P is odd and the oddness of S_{f_2+1} also implies that the term $S_{f_2+1}(c(A))$ is divisible by $c(A)$.

We conclude by proving the result for arbitrary $e \geq 3$, assuming by induction that the result is true for smaller values of e. For this, we split the sum over $b_1, \ldots, b_e$ into two parts:

$$\begin{aligned} P(A) &= \sum_{b_1+\ldots+b_e=c(A)} b_1 \cdots b_e \cdot Q(A,B) \\ &= \sum_{b_1+\ldots+b_{e-2}+\widetilde{b}=c(A)} b_1 \cdots b_{e-2} \cdot \sum_{b_{e+1}+b_e=\widetilde{b}} b_{e+1} b_e \cdot Q(A,B)\,. \end{aligned} \tag{61}$$

By applying the proven case $e = 2$ of the lemma to the auxiliary functions

$$\widetilde{Q}(\underbrace{A, b_1, \ldots, b_{e-2}, \widetilde{b}}_{=\widetilde{A}}, \underbrace{b_{e-1}, b_e}_{=\widetilde{B}}) = Q(A,B),\ \widetilde{c}(\widetilde{A}, \widetilde{B}) = \widetilde{b}\,,$$

we find that

$$F(A, b_1, \ldots, b_{e-2}, \widetilde{b}) = \sum_{b_{e+1}+b_e=\widetilde{b}} b_{e+1} b_e \cdot Q(A,B)$$

is an odd polynomial in the entries of $A, B, \widetilde{b}$, which is divisible by $\widetilde{c} = \widetilde{b}$. In particular we can write it as $F = \widetilde{b} \cdot \overline{F}$ for an even polynomial $\overline{F}$. Plugging this expression back into (61) we find

$$P(A) = \sum_{b_1+\ldots+b_{e-2}+\widetilde{b}=c(A)} b_1 \cdots b_{e-2} \cdot \widetilde{b} \cdot \overline{F}(A, b_1, \ldots, b_{e-2}, \widetilde{b})\,.$$

But this sum is now covered by the proven case of the lemma for $e - 1$ and all desired properties of P follow. □

Proof of Lemma A.1. For the proof, we group the summands (Γ, I) in (17) according to the underlying graph Γ and show that each partial sum is a polynomial in a. Such a graph Γ is specified by the data of

- the number $e \geq 1$ of its edges,
- a partition $g - e + 1 = g' + g''$ of the remaining genus and
- a partition $J' \sqcup J'' = \{1, \ldots, n\}$ of the marked points.

In order to have a nonzero contribution, the markings s, t must go to different vertices, and below we assume that $s \in J'', t \in J'$. Fixing such a graph Γ there are two possible level-assignments $\Gamma^{\pm}$ on Γ, depending on the choice of the vertex of genus g' going to level 0 (in Γ^+) or level -1 (in Γ^-). We claim that depending on the input vector a, at most one of the two orientations can give a nonzero contribution to the sum (17). Indeed, denote by $c : \mathbb{Q}^n \to \mathbb{Q}$ the linear function defined by

$$c(a) = k(2g' - 2 + n') - \sum_{j \in J'} a_j \text{ for } k = \frac{\sum_{i=1}^n a_i}{2g - 2 + n} \text{ and } n' = |J'| + e\,. \tag{62}$$

In Proposition 3.1 we only consider a such that the k defined above is an integer, and then in the formula (17) for $c(a) \geq 0$ we only see contributions (Γ, I) with underlying level-graph Γ^+ (similarly for Γ^- and $c(a) \leq 0$). For $c(a) \geq 0$, the possible twists I on Γ^+ are enumerated by partitions $b_1 + \ldots + b_e = c(a)$ for positive integers b_i, and the sum of all contributions from graph Γ is given by

$$P^+(a) = \sum_{b_1+\ldots+b_e=c(a)} \frac{1}{e!} b_1 \cdots b_e \cdot \zeta_{\Gamma *}\left(\mathrm{DR}_{g'}(a_{J'}, \underline{b}), \otimes \mathrm{DR}_{g''}(a_{J''}, -\underline{b})\right)\,. \tag{63}$$

Here $\underline{b} = (b_1, \ldots, b_n)$ and again we run through the partitions of $c(a)$ with the order of the summands taken into account. Compared to the original formula this is compensated by the fact that we divide by the size $e!$ of the full automorphism group of Γ instead of the group of automorphisms fixing a given twist. We also note that by our conventions of $s \in J'', t \in J'$ we have $f_{s,t}(\Gamma, I) = 1$ here. On the other hand, at points a with $c(a) \leq 0$ the sum of contributions is given by

$$P^-(a) = -\sum_{b_1+\ldots+b_e=-c(a)} \frac{1}{e!} b_1 \cdots b_e \cdot \zeta_{\Gamma *}\left(\mathrm{DR}_{g'}(a_{J'}, -\underline{b}), \otimes \mathrm{DR}_{g''}(a_{J''}, \underline{b})\right) . \tag{64}$$

Using the fact that the double ramification cycle is an even polynomial in its entries (i.e. that $\mathrm{DR}_{g_i}(-A) = \mathrm{DR}_{g_i}(A)$, as follows from its formula or from Invariance I of [BHP+20]) we can apply Lemma A.3 to conclude that both P^+ and P^- are given by polynomials on their respective half-spaces $\{a : c(a) \geq 0\}$ and $\{a : c(a) \leq 0\}$. Moreover, using again that the DR_{g_i}-cycles are even, one sees from (63) and (64) that $P^-(-a) = -P^+(a)$. Combining this with the fact that P^- is odd by Lemma A.3, i.e. $P^-(-a) = -P^-(a)$, it follows that $P^+ = P^-$. Thus indeed the contribution of Γ to the formula (17) is polynomial everywhere, which concludes the proof of the lemma. □

For the results involving intersection numbers of spin double ramification cycles, we also need the following lemma on sums of polynomials over odd integers.

Lemma A.4. *Let $n, m \geq 0$ be integers and $P \in \mathbb{Q}[x_1, \ldots, x_{n+m}]$ any polynomial. Then there exists a polynomial $Q \in \mathbb{Q}[x_1, \ldots, x_n, a]$, such that for $a \geq 0$ of the same parity as m, we have*

$$Q(x_1, \ldots, x_n, a) = \sum_{\substack{j_1,\ldots,j_m>0 \text{ odd}, \\ j_1+\ldots+j_m=a}} P(x_1, \ldots, x_n, j_1, \ldots, j_m) . \tag{65}$$

Moreover, for m odd and all terms of P having of odd total degree in $j_1, \ldots, j_m$, the polynomial Q is divisible by a.

Proof. By decompositing P into monomials and drawing out the factors $x_1, \ldots, x_n$, it is easy to reduce to the case $n = 0$ and P being a monomial of some degree e. Then, we begin with a preparatory remark: using similar techniques as in the proof of Lemma A.3, or alternatively a suitable version of Ehrhart reciprocity, it is possible to show that for any polynomial $P' \in \mathbb{Q}[x_1, \ldots, x_m]$ the assignment

$$\mathbb{Z} \to \mathbb{Q}, b \mapsto \begin{cases} \sum_{\substack{i_1,\ldots,i_m \geq 0, \\ i_1+\ldots+i_m=b}} P'(i_1, \ldots, i_m) & b \geq 0 \\ (-1)^{m-1} \sum_{\substack{i_1,\ldots,i_m<0, \\ i_1+\ldots+i_m=b}} P'(i_1, \ldots, i_m) & b < 0 \end{cases} \tag{66}$$

is given by a polynomial $Q' \in \mathbb{Q}[b]$.

Then, if the original polynomial P is of pure degree e, we can parameterize the odd numbers j_ℓ in (65) as $j_\ell = 2i_\ell + 1$. Using moreover that

$$P(2i_1 + 1, \ldots 2i_m + 1) = 2^e P(i_1 + \frac{1}{2}, \ldots, i_m + \frac{1}{2}) ,$$

we can obtain the sum (65) from (66) by choosing $P'(x_1, \ldots, x_m) = 2^e P(x_1 + 1/2, \ldots, x_m + 1/2)$ and making the substitution b = (a-m)/2. This shows that Q is indeed a polynomial.

Finally, going through the substitutions above one checks that for negative a of the same parity as m, the polynomial Q is given by

$$Q(a) = (-1)^{m-1} \sum_{\substack{j_1,\ldots,j_m<0 \text{ odd},\\ j_1+\ldots+j_m=a}} P(j_1,\ldots,j_m)\,. \tag{67}$$

For m and P odd, it is then immediate that $Q(-a) = -Q(a)$, so that indeed Q is divisible by a. □

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