Proposal for Baby-Seminar: Deformation theory

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1 Main objects of interest

The main object of study are deformation functors. One of the simplest case is the first order deformation. You can think of a first order deformation of a structure S over a field k (a scheme, a Galois representation, ...) to be a structure S' of the same type, defined over the dual numbers $k[\epsilon]/(\epsilon^2)$ in some functorial way and such that it coincides with S when restricted to k. Of course no-one prevents us from considering higher order deformations by considering extensions to $k[\epsilon]/(\epsilon^{n+1})$ rather than only to the dual numbers. Deformation theory studies criteria that S must satisfy in order for such extensions to exist.

There is also a similar set of deformation problems that aim to study lifts of structures over a finite field k to successive artinian quotients of $W(k)$, the ring of Witt vectors of k.

Turns out that the two situation can be approached with quite the same theoretical machinery of deformation over artinian algebras. The typical situation is to fix a field k, say either a finite field (resp. an algebraically closed field), and denote with $W(k)$ its ring of Witt vectors (resp. k itself). We then consider a functor $F : \mathcal{C} \to \text{Sets}$ where $\mathcal{C}$ is the category of complete local artinian $W(k)$-algebras with residue field k and ask if it is **pro-representable**, meaning that it is represented by a complete Noetherian local $W(k)$-algebra R (we say that it is "pro" representable because R might not lie in $\mathcal{C}$. In that case we call R the universal deformation ring or universal family (actually, the family is not literally the ring itself, but rather the collection $\{F(R/\mathfrak{m}^n\}_{n\in\mathbb{N}}$, where $\mathfrak{m}$ is the maximal ideal of R).

2 Plan for the seminar

I sketch a plan:

- The first part of the seminar, after introducing the relevant objects, would be to study criteria for functors as above to be pro-representable. It turns out that often pro-representability is s bit too much to require, but one can relax slightly the condition and study the weaker notions of *miniversal* and *versal* families. There exists so called **Schlessinger's Criteria** that tell when the functor F gives rise to a universal/miniversal/versal family.
- To find applications of the above criteria, we first look at geometry. For example, we know that the Hilbert functor classifying closed subschemes of a projective scheme X over k, say in the case $k = \overline{k}$, is representable. The same holds for the Picard functor representing isomorphism classes of line bundles on X. However the proofs of representability (in the first case at least)

is quite hard. At this point of the seminar we would be able to prove their pro-representability just verifying that the conditions of Schlessinger's Criterion hold in this case. Several other instances could be possibly considered, like lifting of varieties from characteristic zero to characteristic p or deformations of singularities. You can have a look at [Har10] for more examples in this flavour.

- We then turn the attention to Galois representations. The discussion on Schlessinger's Criteria can be found in [Ber+13; CSS13]. After the general discussion, my aim would be to get a feeling of the proof of Taylor-Wiles on the modularity of semistable elliptic curves $E/\mathbb{Q}$ and, consequently, of Fermat Last Theorem. The whole proof in detail is out of reach. However, with the language of deformation theory, we can at least understand the general flavour and some parts in more detail. Several accounts on the overall strategy exists. For example [CSS13] was one of the first published and contains a lot of material.

  The main idea is the following (if you want to read more: I took it from [Ste97], but expositions of the general strategy are more common than Pokemons in the tall grass): an equivalent way of showing that E admits a modular parametrization is to show that the Galois representation $\rho_{E,p} : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to GL_2(\mathbb{Z}_p)$ associated to the Tate module $T_p(E)$ is modular, namely it can be realized in the étale cohomology of some modular curve (whose construction we have seen in the last two talks of the babyseminar on the Weil conjectures in the winter semester). The idea is to start from the representation $\rho_0 := \bar{\rho}_{E,p} : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to GL_2(\mathbb{F}_p)$ corresponding to the p-torsion on E and to define two different deformation functors of ρ_0:

  1. one studying deformations of ρ_0 that look modular, that means that one imposes some conditions that all modular representations and representations of torsion of elliptic curves do satisfy,

  2. the other one studying deformations of ρ_0 that are actually modular.

  Taylor and Wiles show that if one has a semistable elliptic curve E and the representation $\bar{\rho}_{E,3}$ satisfies some conditions, then both (1) and (2) are representable. In fact the deformation ring representing the second problem is a Hecke algebra and they can show that the two deformation rings are isomorphic. Finally the show that if $\bar{\rho}_{E,3}$ does not satisfy the needed properties, then one can deform E to an isogenous curve E' such that $\bar{\rho}_{E,5}$ satisfies the required property. But if E' is modular, then also is E (if the isogeny is $\phi : E' \to E$ you can compose the modular parametrization $X_0(N) \to E'$ with ϕ; if $\phi : E \to E'$ goes the other way around, you can use the dual isogeny.

There are also more general results nowadays, such as the proof of Serre's conjectures by Khare-Winterberg, but they are even more difficult, and probably they don't fit as well in the program of a babyseminar.

References

[Ber+13] Laurent Berger et al. *Elliptic curves, Hilbert modular forms and Galois deformations.* Springer Science & Business Media, 2013.

[CSS13] Gary Cornell, Joseph H Silverman, and Glenn Stevens. *Modular forms and Fermat's last theorem.* Springer Science & Business Media, 2013.

[Har10] Robin Hartshorne. *Deformation theory.* Vol. 257. Springer, 2010.

[Ste97] Glenn Stevens. "An overview of the proof of fermat's last theorem". In: *Modular Forms and Fermat's Last Theorem* (1997), pp. 1–16.