## § The period conjecture

Let us briefly recall the formalism of the period conjecture.

Fix a subfield $k$ of $\mathbb{C}$. Let $\mathcal{C}$ be an abelian category together with two faithful exact functors $T_1, T_2 : \mathcal{C} \to k$-vector spaces. Suppose that there exists an isomorphism $\varphi : T_1 \otimes \mathbb{C} \to T_2 \otimes \mathbb{C}$ of functors $\mathcal{C} \to \mathbb{C}$-vector spaces.

<u>Def</u>: 1. The space of formal periods of $\mathcal{C}, T_1, T_2$, denoted by $\widetilde{\mathbb{P}}(\mathcal{C}, T_1, T_2)$ or $\widetilde{\mathbb{P}}(\mathcal{C})$, is the $k$-vector space generated by the symbols $(p, \omega, \sigma)$ with $p \in \mathcal{C}$, $\omega \in T_1 p$, $\sigma \in (T_2 p)^\vee$ with relations:

(a) bilinearity in $\omega$ and $\sigma$;

(b) functoriality: if $f : p \to p'$ is a morphism in $\mathcal{C}$, then
$$(p, (T_1 f)(\omega), \sigma) = (p, \omega, (T_2 f)^\vee(\sigma)) \text{ for all } \omega \in T_1 p', \sigma \in (T_2 p)^\vee.$$

2. The evaluation map at $\varphi$:
$$ev_\varphi : \widetilde{\mathbb{P}}(\mathcal{C}, T_1, T_2) \longrightarrow \mathbb{C}, \quad (p, \omega, \sigma) \mapsto \langle \sigma, \varphi(\omega) \rangle.$$

3. The periods of $\mathcal{C}, T_1, T_2, \varphi$ are the image of the evaluation map at $\varphi$. They are denoted by $\mathbb{P}(\mathcal{C}, T_1, T_2, \varphi)$ or simply $\mathbb{P}(\mathcal{C})$.

<u>Remark</u>: If $\mathcal{C} = \mathcal{C}(D, T)$ is the diagram category of some diagram $D$ with respect to a representation $T_1 = T$, we can give an analogous description of $\widetilde{\mathbb{P}}(\mathcal{C})$ by only using the vertices and the edges of the diagram $D$. In this case, $\widetilde{\mathbb{P}}(\mathcal{C}(D,T)) = \widetilde{\mathbb{P}}(D)$.

Advantage: replacing $\mathcal{C}(D,T)$ by $D$ gives a much smaller presentation of $\widetilde{\mathbb{P}}(\mathcal{C}(D,T))$ as we can use fewer generators and relations.

We will always turn to the presentation induced by the diagram when possible, as morphisms in the diagram category as difficult to describe.

<u>Def</u>: (Period conjecture for $\mathcal{C}, T_1, T_2, \varphi$) $PC(\mathcal{C})$

The evaluation map $ev_\varphi$ is injective.

"O sol che sani ogni vista turbata,
tu mi contenti sì quando tu solvi,
che, non men che saver, dubbiar m'aggrata.

Dante, Inf., XI, 91-93

<u>Classical period conjecture:</u> (still open!)

$\mathcal{C} = \mathrm{MM}_{\mathrm{Nori}}(k)$ , $T_1 = H^*_{dR}$ , $T_2 = H^*_{sing}$ , $\varphi$: comparison isomorphism

$\mathrm{PC}(\mathrm{MM}_{\mathrm{Nori}}(k))$ : the evaluation map $\widetilde{\mathbb{P}}(\mathrm{MM}_{\mathrm{Nori}}(k)) \to \mathbb{C}$ is injective

Explicitly: The periods $P(\mathrm{MM}_{\mathrm{Nori}}(k)) \subseteq \mathbb{C}$ are isomorphic to the k-vector space generated by $(X, Y, i, \omega, \sigma)$ where $(X, Y, i) \in \mathrm{Pairs}$ , $\omega \in H^i_{dR}(X,Y)$ , $\sigma \in H^{sing}_i(X,Y)$ with relations:

(A): bilinearity in $\omega$ and $\sigma$ ;

(B1): for all $f: X \to X'$ with $f(Y) \subseteq Y'$ $(X, Y, i, f^*\omega, \sigma) = (X', Y', i, \omega, f_*\sigma)$ ;

(B2): for $Z \hookrightarrow Y \hookrightarrow X$ $(Y, Z, i, \omega, \partial\sigma) = (X, Y, i+1, \partial\omega, \sigma)$

<u>Period Conjecture for 1-motives</u> (proved for $k = \overline{\mathbb{Q}}$)

Three equivalent formulations:

•) $\mathrm{PC}(\mathrm{Pairs}^1)$, where $\mathrm{Pairs}^1$ is the full subdiagram of pairs with vertices of the form $(X, Y, i)$ with $i \in \{0,1\}$

<u>Explicitly:</u>

The periods $P(\mathrm{Pairs}^1) \subseteq \mathbb{C}$ are isomorphic to the k-vector space generated by $(X, Y, i, \omega, \sigma)$ where $(X, Y, i) \in \mathrm{Pairs}^1$ , $\omega \in H^1_{dR}(X,Y)$ , $\sigma \in H^{sing}_1(X,Y)$ with relations:

(A'): bilinearity in $\omega$ and $\sigma$ ;

(B1'). for all $f: X \to X'$ with $f(Y) \subseteq Y'$ $(X, Y, i, f^*\omega, \sigma) = (X', Y', i, \omega, f_*\sigma)$ ;

(B2'): for $Z \hookrightarrow Y \hookrightarrow X$ $(Y, Z, 0, \omega, \partial\sigma) = (X, Y, 1, \partial\omega, \sigma)$.

•) $\mathrm{PC}(\mathrm{Curves})$, where curves is the full subdiagram of Pairs with vertices of the form $(C, D, 1)$ with $C$ a smooth affine curve and $D$ a finite collection of points of $C$.

<u>Explicitly:</u>

The periods $P(\mathrm{Curves}) \subseteq \mathbb{C}$ are isomorphic to the k-vector space generated by $(C, D, 1, \omega, \sigma)$ where $(C, D, 1) \in \mathrm{Curves}$ , $\omega \in H^1_{dR}(C,D)$ , $\sigma \in H^{sing}_1(C,D)$ with relations:

(A''): bilinearity in $\omega$ and $\sigma$ ;

(B'') : for all $f: C \to C'$ with $f(D) \subseteq D'$ $(C, D, 1, f^*\omega, \sigma) = (C', D', 1, \omega, f_*\sigma)$.

•) $\mathrm{PC}(1\text{-}\mathrm{Mot}_k)$, where $1\text{-}\mathrm{Mot}_k$ is the category of iso-1-motives.

<u>Explicitly</u>: Boh!

Since we know that $d_1\mathcal{MM}_{Nori}(k) = \mathcal{C}(\text{Pairs}^1, H^*_{sing}) = \mathcal{C}(\text{Curves}, H^*_{sing})$ and it is anti-equivalent to the category of iso-1-motives, the three versions of the period conjecture for 1-motives are equivalent.

$PC(1\text{-}Mot_k)$ is the version that we have proved in the previous talk.

$PC(\text{Pairs}^1)$ and $PC(\text{Curves})$ have the advantage of giving an explicit description of the space of 1-periods via generators and relations.

Let us revisit very briefly the strategy of the proof of the period conjecture for 1-motives.

A key result was the following: ( $k = \overline{\mathbb{Q}}$ from now on)

<u>Theorem 1</u>: Let $M \in 1\text{-}Mot_k$, $\sigma \in V_{sing}(M)$, $\omega \in V_{dR}^\vee(M)$.

Suppose that $\int_\sigma \omega = 0$.

Then there is a short exact sequence $0 \to M_1 \xrightarrow{i} M \xrightarrow{p} M_2 \to 0$ of 1-motives and $\sigma_1 \in V_{sing}(M)$, $\omega_2 \in V_{dR}^\vee(M_2)$ such that $\sigma = i_*\sigma_1$ and $\omega = p^*\omega_2$.

Theorem 1 is a rather direct consequence of the analytic subgroup theorem.

Let us now suppose that $\sum_{i=1}^n a_i \int_{\sigma_i} \omega_i = 0$, where $a_i \in \overline{\mathbb{Q}}$, $\sigma_i \in V_{sing}(M_i)$, $\omega_i \in V_{dR}^\vee(M_i)$ for some $M_1, \dots, M_n \in 1\text{-}Mot_{\overline{\mathbb{Q}}}$.

<u>First step</u>: By pulling back to $M_1 \oplus \dots \oplus M_n$ we can reduce to the case $n=1$, i.e. to the case of $\int_\sigma \omega = 0$ over some 1-motive $M$.

<u>Second step</u>: By applying Theorem 1, we obtain $\int_\sigma \omega = \int_{i_*\sigma_1} p^*\omega_2 = \int_{\sigma_1} i^* p^* \omega_2 = \int_{\sigma_1} 0$, which is trivially zero.

The goal of this talk is the following:

[...] " Se tu segui tua stella,
non puoi fallire a glorïoso porto,
se ben m' accorsi ne la vita bella.

Dante, Inf., XV, 55-57

Let us pick $M \in 1\text{-}\mathrm{Mot}_{\overline{\mathbb{Q}}}$, $\sigma \in V_{\mathrm{sing}}(M)$, $\omega \in V_{\mathrm{dR}}^\vee(M)$. We wish to make the definition of the period $\int_\sigma \omega$ explicit. Write $M = [L \to G]$.

First, $\omega \in V_{\mathrm{dR}}^\vee(M) = \mathrm{Lie}(M^\natural)^\vee = \mathrm{coLie}(M^\natural)$. Since $M^\natural$ is an algebraic group, every cotangent vector at the identity element defines an invariant global differential form on $M^\natural$ (we take this cotangent vector around by using translation maps). Moreover, every invariant global differential 1-form arises in this way. Thus, $\mathrm{coLie}(M^\natural) = \Omega^1_{M^\natural/\overline{\mathbb{Q}}}(M^\natural)^{M^\natural}$.

Thus, we may regard $\omega$ as an actual differential form on $M^\natural$.

Let us now turn to $\sigma \in V_{\mathrm{sing}}(M) = T_{\mathrm{sing}}(M) \otimes_{\mathbb{Z}} \mathbb{Q}$. Recall that we have a pull-back diagram

$$\begin{tikzcd} T_{\mathrm{sing}}(M) \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & \mathrm{Lie}(G^{an}) \arrow[d, "\exp"] \\ L \arrow[r] & G^{an} \end{tikzcd}$$

We had seen that this pullback coincides with the one over $M^{\natural, an}$, i.e.

$$\begin{tikzcd} T_{\mathrm{sing}}(M) \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & \mathrm{Lie}(M^{\natural, an}) \arrow[d, "\exp"] \\ L \arrow[r] & M^{\natural, an} \end{tikzcd}$$

Thus, $T_{\mathrm{sing}}(M)$ is a lattice in $\mathrm{Lie}(M^{\natural, an})$. Up to clearing out denominators, we may assume that $\sigma \in T_{\mathrm{sing}}(M)$, and so $\sigma$ maps to $M^{\natural, an}$ via the exponential map.

Recall that $\exp$ is the universal covering of $M^{\natural, an}$. Thus, $\sigma \in \mathrm{Lie}(M^{\natural, an})$ corresponds to a path in $M^{\natural, an}$ from $0$ to $\exp(\sigma)$.

The period $\int_\sigma \omega$ is precisely the integral of the invariant differential form $\omega$ along this path from $0$ to $\exp(\sigma)$.

<u>Example</u>: Every algebraic number appears as a period of the motive $M = [\mathbb{Z} \to 0]$.
Indeed, $M^\natural = \mathbb{G}_a$, so $V_{\mathrm{sing}}(M) = \mathbb{Q}$ and $V_{\mathrm{dR}}^\vee(M) = \mathrm{coLie}(\mathbb{G}_a) = \overline{\mathbb{Q}}\, dt$. The period map sends $1$ to $\frac{d}{dt}$. Thus, the periods of $M$ are precisely $\overline{\mathbb{Q}}$. In terms of integration, every such period can be written as $\int_0^u \alpha\, dt$ for some $\alpha, u \in \overline{\mathbb{Q}}$.

More generally, it is easy to see that all periods of motives of the form $[L \to 0]$ are algebraic.

## § A transcendence criterion

O Muse, o alto ingegno, or m'aiutate;
o mente che scrivesti ciò ch'io vidi,
qui si parrà la tua nobilitate.

Dante, Inf., II, 7-9

In order to give transcendence proofs, we will mainly exploit the following criterion

<u>Theorem 2</u>: Let $M = [L \to G] \in 1\text{-Mot}_{\bar{\mathbb{Q}}}$, $\sigma \in V_{sing}(M)$, $\omega \in V_{dR}^{\vee}(M)$. Then the period $\alpha = \int_\sigma \omega$ is algebraic if and only if there are $\varphi, \psi \in V_{dR}^{\vee}(M)$ such that

- ·) $\omega = \varphi + \psi$;
- ·) $\int_\sigma \psi = 0$;
- ·) the image of $\varphi$ in $V_{dR}^{\vee}(G)$ vanishes.

<u>Proof</u>: $\Leftarrow$) From the short exact sequence $0 \to [0 \to G] \to [L \to G] \to [L \to 0] \to 0$ we get
$0 \to V_{dR}^{\vee}([L \to 0]) \to V_{dR}^{\vee}(M) \to V_{dR}^{\vee}([0 \to G]) \to 0$.
Suppose that the image of $\varphi$ in $V_{dR}^{\vee}([0 \to G])$ vanishes, i.e. $\varphi$ lies in $V_{dR}^{\vee}([L \to 0])$
We also have a short exact sequence
$0 \to V_{sing}([0 \to G]) \to V_{sing}([L \to G]) \to V_{sing}([L \to 0]) \to 0$,
so we can consider the image $\bar{\sigma}$ of $\sigma$ in $V_{sing}([L \to 0])$. Then $\alpha = \int_\sigma \omega - \int_\sigma \varphi = \int_{\bar{\sigma}} \varphi$ is a period of the motive $[L \to 0]$. This must be algebraic, by the example that we have seen before.

$\Rightarrow$) Assume that $\alpha = \int_{\bar{\sigma}} \omega$ is algebraic. If $\alpha = 0$, we may take $\psi = \omega$ and we are done. Suppose $\alpha \neq 0$.
Since $\alpha$ is algebraic, we may realize it as a period of $[\mathbb{Z} \to 0]$, say $\alpha = \int_{\sigma'} \omega'$ where $\omega' = \alpha\, dt$ and $\sigma' \in V_{sing}([\mathbb{Z} \to 0])$ corresponds to the standard basis vector.
Now consider the motive $M \oplus [\mathbb{Z} \to 0]$, on which we have $\Omega = (\omega, \omega') \in V_{dR}^{\vee}(M \oplus [\mathbb{Z} \to 0])$ and $\Sigma = (\sigma, -\sigma') \in V_{sing}(M \oplus [\mathbb{Z} \to 0])$. Thus $\int_\Sigma \Omega = \int_\sigma \omega - \int_{\sigma'} \omega' = \alpha - \alpha = 0$.
By Theorem 1, we may find a short exact sequence $0 \to M_1 \xrightarrow{(i,j)} M \oplus [\mathbb{Z} \to 0] \xrightarrow{p+q} M_2 \to 0$ of 1-motives together with $\sigma_1 \in V_{sing}(M_1)$, $\omega_2 \in V_{dR}(M_2)$ such that $(i,j)_* \sigma_1 = (\sigma, \sigma')$, $p^* \omega_2 = \omega$, $q^* \omega_2 = \omega'$.
Observe that $\omega' \neq 0$ because $\alpha \neq 0$, so the pullback of $\omega_2$ via $q : [\mathbb{Z} \to 0] \to M_2$ is not zero.
In particular, the map $q : [\mathbb{Z} \to 0] \to M_2$ cannot be the zero morphism. We claim that this implies that $[\mathbb{Z} \to 0]$ is a direct summand of $M_2$. Indeed, write $M_2 = [L_2 \to G_2]$.

The composition $[\mathbb{Z}\to 0]\to[L_2\to G_2]\to[L_2\to 0]$ is non-trivial. Thus, the morphism $\mathbb{Z}\to L_2$ does not vanish. Hence, we may write $L_2$ as the direct sum of the image of $\mathbb{Z}$ in $L_2$ and a direct complement, up to isogeny. This defines a section of $[L_2\to G_2]\to[L_2\to 0]$.

Let us then write $M_2=[\mathbb{Z}\to 0]\oplus M_2'$. We may then write $\omega_2=\varphi_2+\psi_2$ where $\varphi_2\in V_{dR}^\vee([\mathbb{Z}\to 0])$ and $\psi_2\in V_{dR}^\vee(M_2')$. Define $\varphi=p^*\varphi_2$ and $\psi=p^*\psi_2$, which lie in $V_{dR}^\vee(M)$. Since $\omega=p^*\omega_2$, we have $\omega=\varphi+\psi$, so $\alpha=\int_\sigma\omega=\int_\sigma\varphi+\int_\sigma\psi$.

We can split off the direct summand $[\mathbb{Z}\to 0]$ from the above sequence to obtain a short exact sequence $0\to M_1\xrightarrow{i}M\to M_2'\to 0$.

Recall that $\sigma=i_*\sigma'$, so $\int_\sigma\psi=\int_{i_*\sigma'}p^*\psi_2=\int_{\sigma'}i^*p^*\psi_2=\int_{\sigma'}0=0$.

On the other hand, the composition $M\to M_2'\oplus[\mathbb{Z}\to 0]\to[\mathbb{Z}\to 0]$ factors through $M\to[L\to 0]$, so $\varphi=p^*\varphi_2$ is in the image of $V_{dR}^\vee([\mathbb{Z}\to 0])\to V_{dR}^\vee([L\to 0])\to V_{dR}^\vee(M)$.

Thus, the image of $\varphi$ in $V_{dR}^\vee(G)$ vanishes. □

## § Transcendence of $\varpi$

Qual è il geometra che tutto s' affigge
per misurar lo cerchio, e non ritrova,
pensando, quel principio ond' elli indige,

tal era io a quella vista nova:
veder voleva come si convenne
l' imago al cerchio e come vi s' indova;

Dante, Par., XXXIII, 133-138

Consider the 1-motive $M=[0\to\mathbb{G}_m]$. Since every vector extension of $\mathbb{G}_m$ is trivial, we have $M^{\natural}=\mathbb{G}_m$. Thus, $V_{dR}^\vee(M)=\mathrm{coLie}(\mathbb{G}_m)=\Omega^1(\mathbb{G}_m)^{\mathbb{G}_m}$ and $V_{sing}(M)=\ker(\exp:\mathbb{C}\to\mathbb{C}^\times)$.

Take $\omega=\frac{dz}{z}\in V_{dR}^\vee(M)$ and the unit circle $\gamma:[0,1]\to\mathbb{C}^\times$, $s\mapsto e^{2\varpi i s}$, so $\gamma\in V_{sing}(M)$.

Then $\int_\gamma\omega=2\varpi i$ is a period of $M$.

<u>Theorem 3</u> $\varpi$ is transcendental.

Proof: It suffices to show transcendence of $2\pi i$. Suppose that $2\pi i$ is algebraic. Then, by Theorem 2, we may write $\omega = \varphi + \psi \in V_{dR}^{\vee}(M)$ such that $\int_\sigma \psi = 0$, while the image of $\varphi$ vanishes in $V_{dR}^{\vee}(\mathbb{G}_m)$. However, we observed that $M^{\natural} = \mathbb{G}_m$ and the map $V_{dR}^{\vee}(M) \to V_{dR}^{\vee}(\mathbb{G}_m)$ is an isomorphism. Thus, $\varphi = 0$ and $2\pi i = \int_\sigma \omega = \int_\sigma \psi = 0$, which is a contradiction. $\square$

## § Transcendence of logarithms

We turn to transcendence of logarithm of algebraic numbers.

Take $\alpha \in \overline{\mathbb{Q}}^{\times}$ and consider $\log \alpha$, for some fixed branch of the logarithm. We may write $\log \alpha = \int_\gamma \frac{dz}{z}$ where $\gamma$ is a path in $\mathbb{C}^{\times}$ from $1$ to $\alpha$. The choice of $\gamma$ is determined by the branch of the logarithm.

So, let us reinterpret this integral as a period of a 1-motive.

We define the one motive $M(\alpha) = [\mathbb{Z} \to \mathbb{G}_m]$, where the map $\mathbb{Z} \to \mathbb{G}_m(\overline{\mathbb{Q}})$ sends $1$ to $\alpha$. $M(\alpha)$ is also called "Kummer motive".

Let us study singular and de Rham realization of $M(\alpha)$. We know that

$M(\alpha)^{\natural} = \mathrm{Hom}_{ab}(\mathbb{Z}, \mathbb{G}_a)^{\vee} \times \mathbb{G}_m^{\natural} \cong \mathbb{G}_a \times \mathbb{G}_m$.

Thus $V_{dR}^{\vee}(M(\alpha)) = \mathrm{coLie}(\mathbb{G}_a \times \mathbb{G}_m) = \mathrm{coLie}(\mathbb{G}_a) \oplus \mathrm{coLie}(\mathbb{G}_m)$. We consider $(0, \frac{dz}{z}) \in V_{dR}^{\vee}(M(\alpha))$.

The singular realization is defined by the following pull-back:

$$\begin{tikzcd} T_{sing}(M(\alpha)) \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & \mathbb{C} \arrow[d, "\exp"] \\ \mathbb{Z} \arrow[r] & \mathbb{C}^{\times} \end{tikzcd} \qquad \begin{tikzcd} (n,z) \arrow[r, mapsto] \arrow[d, mapsto] & z \arrow[d, mapsto] \\ n \arrow[r, mapsto] & \alpha^n = e^z \end{tikzcd}$$

Assume that $\alpha$ is not a root of unity.

Then $T_{sing}(M(\alpha)) = \{(n,z) \in \mathbb{Z} \times \mathbb{C} \mid \alpha^n = e^z\} = \{(n, n\log\alpha + 2\pi i k) \in \mathbb{Z} \times \mathbb{C} \mid n, k \in \mathbb{Z}\} \overset{(*)}{\cong}$

$\cong \{ n\log\alpha + 2\pi i k \in \mathbb{C} \mid n,k \in \mathbb{Z}\} = \mathrm{Span}_{\mathbb{Z}}\langle \log\alpha, 2\pi i\rangle = \exp^{-1}(\alpha^{\mathbb{Z}})$.

The isomorphism $(*)$ is justified by the fact that $\log\alpha$ and $2\pi i$ are linearly independent over the rationals (if $n\log\alpha + 2\pi i k = 0$, then $\alpha^n = (-1)^k$, which cannot happen since $\alpha$ is not a root of unity).

If $\alpha$ is a root of unity, then $M(\alpha) \cong \mathbb{G}_m \oplus [\mathbb{Z} \to 0]$. Indeed, the morphism $\mathbb{Z} \to \mathbb{G}_m(\overline{\mathbb{Q}}) = \overline{\mathbb{Q}}^{\times}$, $1 \mapsto \alpha$ fits into a short exact sequence $0 \to N\mathbb{Z} \to \mathbb{Z} \to \overline{\mathbb{Q}}^{\times} \to \overline{\mathbb{Q}}^{\times}/\langle\alpha\rangle \to 0$ in which $N\mathbb{Z} \to \mathbb{Z}$ and $\overline{\mathbb{Q}}^{\times} \to \overline{\mathbb{Q}}^{\times}/\langle\alpha\rangle$ become isomorphisms in the isogeny category.

It follows that $\mathbb{Z} \to \mathbb{G}_m(\overline{\mathbb{Q}})$ is the zero morphism, hence the splitting. Anyway, if $\alpha$

is a root of unity, then $\log\alpha = \frac{n2\varpi i}{m}$ for some $n, m \in \mathbb{Z}$, $m \neq 0$, which we already know to be transcendental. So we will assume that $\alpha$ is not a root of unity.

Thus $V_{sing}(M(\alpha)) = \mathrm{Span}_{\mathbb{Q}}\langle \log\alpha, 2\varpi i\rangle \subseteq \mathbb{C} \cong Lie(\mathbb{G}_m^{an})$.

By recalling that $\exp: Lie(\mathbb{G}_m^{an}) \to \mathbb{G}_m^{an}$ is the universal cover of $\mathbb{G}_m^{an}$, we identify $\log\alpha$ with a path from 1 to $\alpha$ in $\mathbb{C}^\times$, say $\sigma_\alpha$, and $2\varpi i$ with the positively oriented loop around 0 in $\mathbb{C}^\times$, say $\gamma$. Thus, $V_{sing}(M(\alpha)) = \langle \sigma_\alpha, \gamma\rangle_{\mathbb{Q}}$.

To compute the periods, we only need to go through the identifications

$V_{sing}(M(\alpha)) \subseteq Lie(M(\alpha)^\natural) = Lie(\mathbb{G}_a^{an}) \oplus Lie(\mathbb{G}_m) = \mathbb{C} \times \mathbb{C}$

and $\gamma \mapsto (0, \gamma)$, $\sigma_\alpha \mapsto \sigma_\alpha^\natural = (1, \sigma_\alpha)$

Thus $\int_{\sigma_\alpha^\natural} \omega = \int_{\sigma_\alpha} \frac{dz}{z} = \log\alpha$.

<u>Theorem 4</u>: For every algebraic number $\alpha \neq 0, 1$, $\log\alpha$ is transcendental, independently of the choice of the logarithm.

<u>Proof</u>: Since we know that $2\varpi i$ is transcendental, we may assume that $\alpha$ is not a root of unity. Suppose that $\int_{\sigma_\alpha} \omega$ is algebraic. By Theorem 2, we may find a decomposition $\omega = \varphi + \psi$ with $\int_{\sigma_\alpha} \psi = 0$ and $\varphi$ that vanishes in $V_{dR}^\vee(\mathbb{G}_m)$.

Let us have a closer look at $\psi$. Since $\int_{\sigma_\alpha} \psi = 0$, by Theorem 1 we may find a short exact sequence $0 \to M' \xrightarrow{i} M(\alpha) \xrightarrow{p} M'' \to 0$ such that $\sigma_\alpha$ lies in the image of $i_*$ and $\psi$ in the image of $p^*$.

The only subgroups of $\mathbb{G}_m$ up to isogeny are 0 and $\mathbb{G}_m$, so it is easy to check that the only possibilities for $M'$ are $0$, $[0 \to \mathbb{G}_m]$ and $M(\alpha)$ itself. Now notice that $\sigma_\alpha \neq 0$, so we exclude $M' = 0$. Moreover, $V_{sing}([0 \to \mathbb{G}_m])$ maps in $V_{sing}(M(\alpha))$ with image $\langle\gamma\rangle_{\mathbb{Q}}$, which does not include $\sigma_\alpha$. We are only left with $M' = M(\alpha)$, which leads to $M'' = 0$ and $\psi = 0$.

We conclude that $\omega = \varphi$ vanishes in $V_{dR}^\vee[0 \to \mathbb{G}_m]$. But its image is $\frac{dz}{z}$ ... $\square$

Next step: we can ask for linear independence of logarithms of algebraic numbers.

The first question in this direction was Hilbert's seven problem:

Given algebraic numbers $\alpha, \beta \neq 0, 1$ with $\beta$ algebraic, is it true that $\alpha^\beta$ is transcendental?

A positive answer was found independently by Gelfond and Schneider in 1934.

An equivalent reformulation is the following:

<u>Theorem 5</u>: Let $\alpha, \beta$ be algebraic numbers such that $\log\alpha$ and $\log\beta$ are linearly dependent over $\overline{\mathbb{Q}}$. Then $\log\alpha$ and $\log\beta$ are linearly dependent over $\mathbb{Q}$.

<u>Proof</u>: Let $a, b \in \overline{\mathbb{Q}}$ be such that $a\log\alpha + b\log\beta = 0$.

Consider the 1-motive $M = [\mathbb{Z} \to \mathbb{G}_m \times \mathbb{G}_m]$ with structure morphism $\mathbb{Z} \to (\overline{\mathbb{Q}}^\times)^2$, $1 \mapsto (\alpha, \beta)$.

As we have seen above, it is straightforward to check that $M = [\mathbb{Z} \to 0] \oplus [0 \to \mathbb{G}_m^2]$ if both $\alpha$ and $\beta$ are roots of unity. In this case, $\log\alpha$ and $\log\beta$ are rational multiples of $2\pi i$, so the claim is trivial. We assume henceforth that $\alpha$ and $\beta$ are not roots of unity.

First, observe that $M^\natural = \mathrm{Hom}_{ab}(\mathbb{Z}, \mathbb{G}_a)^\vee \times (\mathbb{G}_m^2)^\natural = \mathbb{G}_a \times \mathbb{G}_m^2$.

A basis for $V_{dR}^\vee(M) = \mathrm{coLie}(\mathbb{G}_a \times \mathbb{G}_m^2)$ is given by $(dt, 0, 0), (0, \frac{dz_1}{z_1}, 0), (0, 0, \frac{dz_2}{z_2})$.

Let us consider $\omega = a\frac{dz_1}{z_1} + b\frac{dz_2}{z_2}$.

As before, we wish to find some $\sigma \in V_{sing}(M)$ such that $\int_\sigma \omega = a\log\alpha + b\log\beta = 0$, then apply Theorem 2. We therefore need to study $V_{sing}(M)$.

Recall that $T_{sing}(M)$ is defined by the following pull-back diagram:

$$\begin{tikzcd} T_{sing}(M) \arrow[r] \arrow[d] & \mathbb{C} \times \mathbb{C} \arrow[d, "\exp"] \\ \mathbb{Z} \arrow[r] & \mathbb{C}^\times \times \mathbb{C}^\times \end{tikzcd} \qquad \begin{tikzcd} (n, z_1, z_2) \arrow[r, mapsto] \arrow[d, mapsto] & (z_1, z_2) \arrow[d, mapsto] \\ n \arrow[r, mapsto] & (\alpha^n, \beta^n) = (e^{z_1}, e^{z_2}) \end{tikzcd}$$

Thus $T_{sing}(M) = \{(n, z_1, z_2) \in \mathbb{Z} \times \mathbb{C} \times \mathbb{C} \mid (\alpha^n, \beta^n) = (e^{z_1}, e^{z_2})\} \cong$

$$\cong \{ n(\log\alpha, \log\beta) + k_1(2\pi i, 0) + k_2(0, 2\pi i) \mid n, k_1, k_2 \in \mathbb{Z}\} =$$

$$= \mathrm{Span}_{\mathbb{Z}} \langle (\log\alpha, \log\beta), (2\pi i, 0), (0, 2\pi i) \rangle \subseteq \mathbb{C} \times \mathbb{C}.$$

As $\exp: \mathbb{C} \times \mathbb{C} \to \mathbb{C}^\times \times \mathbb{C}^\times$ is the universal covering of $\mathbb{C}^\times \times \mathbb{C}^\times$, we may identify $(\log\alpha, \log\beta) \in T_{sing}(M)$ with a path $\sigma$ going from $(1,1)$ to $(\log\alpha, \log\beta)$ (the path depending on the choice of the branch of the logarithm). This path will be of the form $\sigma = (\gamma_\alpha, \gamma_\beta)$ where $\gamma_\alpha$ and $\gamma_\beta$ are two paths on $\mathbb{C}^\times$ from $1$ to $\log\alpha$ and $\log\beta$ respectively.

Then $\int_\sigma \omega = a\int_{\gamma_\alpha} \frac{dz_1}{z_1} + b\int_{\gamma_\beta}\frac{dz_2}{z_2} = a\log\alpha + b\log\beta = 0$.

By Theorem 2, there is a short exact sequence $0 \to M_1 \to M \to M_2 \to 0$ in $1\text{-}\mathrm{Mot}_k$ such that $\sigma$ comes from $V_{sing}(M_1)$ and $\omega$ from $V_{dR}^\vee(M_2)$.

Since $M_2$ is a quotient of $M = [\mathbb{Z} \to \mathbb{G}_m^2]$, we must have $M_2 = [\mathbb{Z}^s \to \mathbb{G}_m^t]$ for some $s \in \{0,1\}$, $t \in \{0,1,2\}$.

We cannot have $t=0$, otherwise $\omega = \left(0, \frac{dz_1}{z_1}, \frac{dz_2}{z_2}\right)$ would be the pull-back of a 1-motive of the form $[\mathbb{Z}^s \to 0]$, so the second and third components of $\omega$ would vanish.

On the other hand, if we had $t=2$, then the map $M \to M_2$ is the identity at the level of semiabelian varieties. This implies that the pushforward of $\sigma$ in $M_2$ (which is 0, since $\sigma$ comes from $M_1$) coincides with $(\gamma_\alpha, \gamma_\beta) = \left(\log\alpha \frac{d}{dz_1}, \log\beta \frac{d}{dz_2}\right) \in \mathrm{Lie}\left((\mathbb{G}_m^{an})^2\right)$.

Thus $\log\alpha = 0 = \log\beta$, which implies that $\alpha = \beta = 1$. This case had already been excluded since we are assuming that $\alpha$ and $\beta$ are not roots of unity.

We must therefore have $t=1$. The map $M = [\mathbb{Z} \to \mathbb{G}_m^2] \to M_2 = [\mathbb{Z}^s \to \mathbb{G}_m]$ must consist, at the level of semi-abelian varieties, of a map of the form $\mathbb{G}_m^2 \to \mathbb{G}_m$, $(x,y) \mapsto x^n y^m$ for some $n, m \in \mathbb{Z}$.

At the level of Lie algebras, the induced map $\mathrm{Lie}(\mathbb{G}_m^2) \to \mathrm{Lie}(\mathbb{G}_m)$ sends $\sigma = \left(\log\alpha \frac{d}{dz_1}, \log\beta \frac{d}{dz_2}\right)$ to $(n\log\alpha + m\log\beta)\frac{d}{dz}$. As before, this image must vanish, since $\sigma$ comes from $M_1$. Thus $n\log\alpha + m\log\beta = 0$, as desired. $\square$

By applying this argument to several logarithms, one derives a proof for the most beautiful theorem in Mathematics:

<u>Theorem 6:</u> (Baker, 1967) Let $\alpha_1, \dots, \alpha_n$ be non-zero algebraic numbers.

If $\log\alpha_1, \dots, \log\alpha_n$ are linearly dependent over $\overline{\mathbb{Q}}$, then they are linearly dependent over $\mathbb{Q}$.

More precisely, we have $\mathrm{rk}\langle \alpha_1, \dots, \alpha_n \rangle_{\mathbb{Z}} = \dim_{\overline{\mathbb{Q}}} \langle \log\alpha_1, \dots, \log\alpha_n, 2\pi i \rangle / 2\pi i \overline{\mathbb{Q}}$.

After this seminar, this result should probably sound rather believable to you.

To make the story more intriguing, let me spice it up a little...

*Sempre a quel ver c'ha faccia di menzogna*
*de' l'uom chiuder le labbra fin ch'el puote,*
*però che sanza colpa fa vergogna;*

*ma qui tacer nol posso; [...]*

Dante, Inf., XVI, 124-127

Baker's astonishing achievement was not only to prove a qualitative version of Theorem 6, but rather an effective quantitative version. He could provide lower bounds for the modulus of a linear form in logarithms $\Lambda = a_1 \log \alpha_1 + \dots + a_n \log \alpha_n$ with $a_i \in \mathbb{Q}$, $\alpha_i \in \overline{\mathbb{Q}}^\times$ such that $\log \alpha_1, \dots, \log \alpha_n$ are linearly independent over $\mathbb{Q}$, the bound depending only on the degree and height of $\alpha_1, \dots, \alpha_n$ (and the determinations of the logarithms).
This effective version allowed him to prove, among other things the existence of only finitely many quadratic imaginary fields with class number 1.
This quantitative version, as far as I know, seems to lie outside of this motivic realm.

## § Periods of abelian varieties

<u>Theorem 7</u>: Let $A$ be an abelian variety, $\omega \in \Omega^1_{A/\overline{\mathbb{Q}}}(A)$, $\gamma$ a closed path on $A^{an}$.
Then $\int_\gamma \omega$ is either zero or transcendental.

<u>Proof</u>: Consider the 1-motive $M = [0 \to A]$. Every global differential form on $A$ is $A$-invariant, so $\omega$ defines an element of $\Omega^1_{A/k}(A)^A = \mathrm{coLie}(A) \subseteq \mathrm{coLie}(A^\natural) = V_{dR}^\vee(M)$. Since $M$ is the motive of an algebraic group, we have $V_{sing}(M) = H_1^{sing}(A^{an}, \mathbb{Q})$, so we can view $\gamma$ as an element of $V_{sing}(M)$. Thus $\int_\gamma \omega$ is a period of $M$.
Suppose that $\int_\gamma \omega$ is algebraic. By Theorem 2, we may write $\omega = \varphi + \psi$ with $\int_\gamma \psi = 0$ and $\varphi$ which restricts to zero on the group part of $M$. But $M = [0 \to A]$ consists of its group part only, thus $\varphi = 0$ and $\int_\gamma \omega = \int_\gamma \psi = 0$ □

To give some explicit examples, consider the elliptic curves with affine equations
$y^2 = 4x^3 - 4x$ and $y^2 = 4x^3 - 4$
which are the only elliptic curves with complex multiplication in the ring of integers of $\mathbb{Q}(i)$ and $\mathbb{Q}(\sqrt{-3})$. One can check that numbers arising as fundamental periods of these curves are $\dfrac{\Gamma(1/4)^2}{\sqrt{8\varpi}}$ and $\dfrac{\Gamma(1/3)^3}{\sqrt[3]{16} \cdot \varpi}$ respectively.
It follows that the numbers $\dfrac{\Gamma(1/4)}{\varpi}$ and $\dfrac{\Gamma(1/3)^3}{\varpi}$ are transcendental.

In this direction, there are nowadays much sharper results. For example, by the work of Chudnovsky on the algebraic independence of periods of elliptic curves, it is known that $\Gamma(1/4)$ and $\varpi$ are algebraically independent. The same holds for $\Gamma(1/3)$ and $\varpi$.

However, the period conjecture for 1-motives is only able to detect $\overline{\mathbb{Q}}$-linear relations among periods of curves. In order to study algebraic independence, one would need to allow products of periods, which on the motivic side correspond to taking products of varieties (which therefore no longer have dimension 1). For this, the full version of the period conjecture would be necessary.

## § Conclusions

We have seen how the motivic set-up allows one to give transcendence proofs in a swift and coincise way.

Let me only remark that the essentially analytic apparatus of classical transcendence proofs is only hidden behind the arguments that we have seen in this talk, and it is not simply disregarded by this motivic reinterpretation. Indeed, all the arguments that we have seen today rely heavily on the analytic subgroup theorem, which exploits all the machinery of transcendental number theory.

The main advantage of this motivic reinterpretation is to give a deeper and more structured account on transcendence proofs. Especially when it comes to computing the dimension of period spaces, motives allow to write some coincise formulae which would appear extremely arbitrary otherwise. The general hope is that a better understanding of the motivic side of the story might empower transcendence proofs with geometric insights.

I hope you all have learnt something interesting, new or perhaps even useful in this seminar. If anything, now you can say you have seen a proof of the transcendence of $\varpi$ at least once in your life.

All the best for the next Babyseminar, go and rock!

Non aspettar mio dir né più mio cenno;
libero, dritto e sano è tuo arbitrio,
e fallo fora non fare a suo senno:

perch'io te sovra te corono e mitrio.

Dante, Purg., XXVII, 139-142