Babyseminar: the geometric Satake equivalence

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Let G be a reductive group over $\mathbf{C}$, that is, a complex algebraic group satisfying a tameness condition. (Eg. $\mathrm{GL}_n, \mathrm{SL}_n$ any other "named" group that is not $\mathbf{G}_a^n$). Such objects have been classified and very explicitly so. To each such group one associates a *root data* $(X, \Phi, X^\vee, \Phi^\vee)$ which pins down G (up to isomorphism).

The definition of root data (which won't be recalled here) is symmetric and in particular defines a group $G^\vee$ called the *Langlands dual group* via

$$\begin{array}{ccc} G & \longleftrightarrow & (X, \Phi, X^\vee, \Phi^\vee) \\ \downarrow & & \downarrow \\ G^\vee & \longleftrightarrow & (X^\vee, \Phi^\vee, X, \Phi) \end{array}$$

The functor $G \mapsto G^\vee$ can be very mysterious. For example $\mathrm{GL}_n^\vee = \mathrm{GL}_n$ but the dual of SL_n is PGL_n and the dual of SO_{2n+1} is Sp_{2n}. The following theorem is, to my knowledge, the only concrete characterisation of the dual group that we have.

Theorem. Let $\mathscr{O} = \mathbf{C}[[t]]$ and $K = \mathbf{C}((t))$. There is an equivalence of Tannakian categories

$$\mathsf{Rep}_{G^\vee} \cong \mathrm{Perv}_{G(\mathscr{O})}(G(K)/G(\mathscr{O}))$$

Our goal in this seminar is to make sense of the above theorem and why it is so beautiful. Here is a breakdown of what is going on in there:

- A Tannakian category is a certain abelian category endowed with a tensor structure and a forgetul functor to $\mathbf{C}$-vector spaces[1]. This completely recovers $G^\vee$ in our case as the $\otimes$-endomorphisms of this forgetful functor.

[1] We focus on $\mathbf{C}$-coefficients for this seminar overview, but there are versions with more general coefficients as $\mathbf{F}_q$ and even $\mathbf{Z}$.

- The notion of perverse sheaves is due to Bernstein, Beilinson, Deligne and Gabber, based on previous work of Goresky and MacPherson. You can think of it as a notion of local system which takes the singularities of its support into account.

- I'm being a bit cavalier by writing $G(K)/G(\mathcal{O})$. In fact, one can define a functor $Gr_G \colon A/k \mapsto G(A((t)))/G(A[[t]])$ which is represented by an ind-scheme over k: this is a colimit of k-varieties along closed immersions, and one may imagine it as an infinite dimensional variety. This is called the *affine Graßmanian* of G.[2]

- The notation $\mathrm{Perv}_{G(\mathcal{O})}(G(K)/G(\mathcal{O}))$ is meant to indicate that the sheaves in question are endowed with a certain equivariant structure induced by the action of $G(\mathcal{O})$ on Gr_G (on the left).

The Theorem will follow from a good undestanding of both sides of the equation above. After all the Tannakian formalism tells us that one we prove that the right hand side is Tannakian then it is already equivalent to the representations of a reductive group and we need only to identify which group it is (and hence only its root data).

We then breakup the Theorem into very fun subtasks: understanding representations of reductive groups (they form a semisimple category in this case), understanding the affine Graßmanian with its equivariant structure, and finally, and crucially, understanding the convolution product structure on the right hand side, which turns out to be the main technical difficulty of the proof.

I want again to emphasize that in this seminar we will learn about a bunch of fun mathematical objects that are embedded into modern research. The ideas in here have various generalizations which are too long to list in detail. Some directions are taking G over a local field, or pushing for a "derived" perspective, not to mention the geometric Langlands program which is a vast generalization of the Theorem above.

Finally, I'll end these notes in a terse and unexplained corollary, the classical Satake isomorphism, of which the Theorem is a categorification of. For more details, vote on this seminar!

Corollary (Satake)**.** Let G, $\mathcal{O}$, K be as above and fix a maximal torus, Borel $T \subset B \subset G$ and Weyl group W. Then one has an isomorphism of $\mathbf{C}$-algebras

$$\mathbf{C}[X_*(T)]^W \cong \mathbf{C}_c[G(\mathcal{O})\backslash G(K)/\,G(\mathcal{O})].$$

[2]Here is a fun mental exercise: think of why this is an algebraic version of the space of loops in G up to homotopy! In fact, it was the string theorists that concieved of this idea first.