Normed vector spaces and the tannakian building

thiago solovera e nery

April 13th, 2024

These are the notes for a talk on the Kleine AG, April 2024.

1 Norms

Lets start by fixing a non-archimedian field K and a non-archimedian norm $|\_|: K \to \mathbf{R}$. (The paper fixes an ordered group Γ, but every non-archimedean field has $\Gamma \subset \mathbf{R}$ by definition and we will only need the case where, in fact, $|K| = \mathbf{R}$.) Let $A \subset K$ be its ring of integers and k its residue class field.

Definition 1. Let V be a K-vector space. A non-archimedian norm on K is a function $|\_|: V \to \mathbf{R}$ such that

1. $|x| = 0$ if and only if $x = 0$;

2. $|\lambda x| = |\lambda||x|$;

3. $|x + y| \leqq \max\{|x|, |y|\}$.

A normed vector space is a K-vector space endowed with a norm. A linear map $V \to V'$ is said to be contractive if $|f(x)| \leq |x|$ for all $x \in V$. In particular they are continuous.

If $W \subset V$ is a subspace, then it inherits a norm. The quotient V/W inherits the norm

$$|v + W| = \min_{w \in W} |v + w|$$

and becomes normed also, and the map $V \to V/W$ is strict. A homomorphism $f: V \to V'$ is called strict if it induces an isomorphism of normed vector spaces $V/\ker f \xrightarrow{\sim} f(V) \subset V'$.

If $V = K$ then one sees that the usual norm is a norm in the sense above. Furthermore, point 2 above implies that the choice of a norm on V is determined by choosing any point $v \in V$ and $|v| \in \mathbf{R}$. In higher dimensions, such argument breaks down, and we make it into a definition.

Definition 2. Let V be a normed vector space. A splitting of V is defined to be a K-basis e_i such that

$$\left|\sum_i \lambda_i e_i\right| = \max_i |\lambda_i||e_i|.$$

The terms splittable, split and etc are defined as usual.

Remark. If K is spherically complete (eg. a DVF) then every K vector space is splittable. In fact the converse is also true, as one may find in that case an extension $L \supset K$ with $|L| = |K|$ and $k = l$. Taking $a \in L - K$ the vector space $K \oplus aK$ with obvious norm does not admit a splitting.

Definition 3. Let V be a normed vector space over K. For $\lambda \in \mathbf{R}_{>0}$ we define

$$V^{\leqq\lambda} = \{v \in V \mid |v| \leqq \lambda\}, \qquad V^{<\lambda} = \{v \in V \mid |v| < \lambda\}.$$

Clearly each $V^{\leqq\lambda}$ is a flat A-module, and a lattice in V if finitely generated. Furthermore if x is in K then

$$V^{\leqq\lambda} \xrightarrow{\sim} V^{\leqq|x|\lambda}$$

via multiplication by x.

The $\{V_\lambda\}_{\lambda\in\mathbf{R}}$ therefore determine a filtration on V (which for example makes V into a topological vector space in a natural way). The associated graded

$$\mathrm{gr}(V) = \oplus_{\lambda\in\mathbf{R}} V^{\leqq\lambda}/V^{<\lambda}$$

is therefore a k-vector space. We note that $K^\times$ acts on V compatibly with the Γ-filtration in the sense that $x \in K^\times$ maps $V^{\leqq\lambda}$ into $V^{\leqq|x|\lambda}$. This also descends to an action of $K^\times$ on $\mathrm{gr}(V)$.

Definition 4. Let K be a non-archemedean field. We denote by $\mathrm{Norm}^\circ(K)$ the category of splittable normed finite dimensional K-vector spaces. The morphisms are contractible (but not necessairly strict) homomorphisms of such.

We endow this category with some structure. Firstly, we see that this is an exact A-linear category, with short exact sequences given by strict extensions. Secondly, there is a canonical forgetful faithful functor

$$\mathrm{forg}\colon \mathrm{Norm}^{\circ}(K) \to \mathrm{Vect}(K)$$

forgetting the underlying norm. Similarly, we have a forgetul functor

$$(\_)^{\leqq 1}\colon \mathrm{Norm}^{\circ}(K) \to \mathrm{Mod}(A)$$

given by $V \mapsto V^{\leqq 1}$ (and similarly for $\leqq x$). This is also faithful since $f(x) = \pi^n f(x/\pi^n)$.

Most importantly, $\mathrm{Norm}^{\circ}(K)$ is a strict tensor category. Firstly, we have a tensor product $V \otimes W$ where we define

$$|z| = \min_{z=\sum_j v_j \otimes w_j} \max_j |v_j||w_j|.$$

Here we check that this is a well defined norm and that if e_i and f_j split V and W we have that $e_i \otimes f_j$ splits $V \otimes W$.

For the duals, we define a norm on $V^* = \mathrm{Hom}(V, K)$ as

$$|\phi| = \max\{|\phi(v)|/|v| \mid v \neq 0\}$$

and check that we have maps $V \otimes V^* \to K$ and $K \to V \otimes V^*$ as usual by choosing a splitting basis and showing its independent of it. We can now also re-interpret the A-linear structure as

$$\mathrm{Hom}(V, W) = (V^* \otimes W)^{\leqq 1} \in \mathrm{Mod}(A).$$

1.1 Lattices

A lattice in a K vector space V is an A-submodule $L \subset V$ such that L is finitely generated and

$$L \otimes_A K \xrightarrow{\sim} V.$$

Since L is automatically flat, this is equivalent to L being (finitely locally) free.

Definition 5. Let $L \subset V$ be a lattice in a finite dimensional K-vector space. We define a norm $|\_|_L = |\_|$ associated to L to be

$$|v| = \left|\sum_i \lambda_i v_i\right| = \max_i |\lambda_i|$$

for $\{v_i\}$ A-basis of L. Clearly this is a splittable norm with $V^{\leqq 1} = L$ and $|V| = |K|$.

Proposition 1. Let V be a splittable normed K-vector space, and suppose that $|V| \subset |K|$. Then $V^{\leqq 1}$ is a lattice and this induces the norm on V.

Proof. If V is a normed K-vector space, $\{e_i\}$ splits V, and $|e_i| \in |K|$ (and hence we can assume that $|e_i| = 1$), then

$$L = e_1 A \oplus \cdots \oplus e_d A = V^{\leqq 1}$$

is a lattice and the norm comes from it by its non-archimedeanness. □

Corollary 1. *If* $|K| = \mathbf{R}$ *then lattices are in bijection with splittable norms.*

We are interested, however, mostly in the case where K is a discretely valued field. In that case, it is not enough to remember a lattice, but we must also remember an $\mathbf{R}$-filtration in a compatible way. (This is indeed, included in the original definition of the building).

Definition 6. Let V be a finite dimensional K-vector space. Suppose given an A-lattice $L \subset V$ and an $\mathbf{R}$-grading χ of L, ie.

$$L = \bigoplus_{w \in \mathbf{R}} L_w, \quad (V = \bigoplus_{w \in \mathbf{R}} V_w)$$

Then we define a norm $|\_| = |\_|_{L,\chi}$ via

$$|v| = \left|\sum_w v_w\right| = \max_w w.|v_w|_w \in \mathbf{R}$$

where $|\_|_w$ is the norm on V_w defined by L_w.

Remark. A grading of L as above can be seen as the same data as a homomorphism

$$\chi\colon D_A \to \mathsf{GL}(L) \cong \mathsf{GL}_{d,A},$$

where D_A is the "diagonal" A-group scheme given by the Hopf group-algebra $A[\mathbf{R}]$. (Its representation parametrize $\mathbf{R}$-gradings on A-modules).

Proposition 2. The norms $|\_|_{L,\chi}$ are splittable for every L and χ as above. Every splittable norm is furthermore of this form (in more than one way).

2 Normed Fiber Functors

We now fix a smooth affine model $\mathcal{G}$ of G over A. Also fix some non-archimedean extension L/K with integral elements B and residue l. (Not required to be algebraic, or to preserve the value group.)

Definition 7. A normed fiber functor over some non-archimedean extension L/K (with respect to our fixed model $\mathcal{G}$) is an A-linear tensor exact (faithful) functor

$$\alpha\colon \mathsf{Rep}^\circ(\mathcal{G}) \to \mathsf{Norm}^\circ(L)$$

from the category of dualizable $\mathcal{G}$-representations in finite (free) A-modules to the category of L-norms as defined last section.

Definition 8. Let $\omega\colon \mathsf{Rep}^\circ(\mathcal{G}) \to \mathsf{Vect}_K$ be a fiber functor. We define the set of norms on ω, $N^\otimes(\omega)$, to be the set of normed fiber functors whose underlying fiber functor is ω. We also define $N^\otimes(\mathcal{G})$, the set of norms of $\mathcal{G}$, to be the set of norms on the standard (forgetful followed by base-change) fiber functor of $\mathcal{G}$.

This comes equipped with two actions

- An $\mathbf{R}$-action given by the canonical $\mathbf{R}$-action on the norms.
- A $G(K)$-action given by the canonical action of $G(K)$ on $L \otimes K$.

For clarity, if $|\_|\colon V \to \mathbf{R}$ is some norm, then so is $\lambda|\_|$ for $\lambda \in \mathbf{R}_{>0} \cong \mathbf{R}$. Similarly, if $T \in \mathsf{GL}(V)$ then so is $|T^{-1}\_|$.

The goal of this seminar is to show that in fact, this is nothing but the (extended) Bruhat-Tits building of $G = \mathcal{G}_K$. The more modest goal of this talk, is to establish the result for split tori.

For now, we show that the building is non-empty, and in fact has a canonical base point. Consider the standard integral fiber functor

$$\lambda\colon \mathsf{Rep}^{\circ}(\mathcal{G}) \to \mathsf{Mod}_A$$

given by forgetting the action. This determines a canonical lattice $\lambda(V) \subset \lambda(V) \otimes K = \omega(V)$. In particular it determines a splittable norm on it.

2.1 Example: the building of a torus

Let Γ be an abelian group. There is an associated algebraic group over $\mathbf{Z}$

$$D^{\Gamma} = \mathsf{Spec}\,\mathbf{Z}[\Gamma]$$

given by the Hopf algebra above. From a tannakian perspective, D^{Γ}-representations are given by a Γ-grading on a lattice. (The γ-graded part being associated with the γ-eigenspace for such representation.)

Very crucially, we may identify $D^{\mathbf{Z}}$ with $\mathbf{G}_m$ and $D^{\mathbf{Z}^n}$ with $\mathbf{G}_m^n$. For a ring A we write D^{Γ}_A for the base change of the group scheme above to $\mathsf{Spec}\,A$.

Lemma 1. Fix a field k, and let T be a split k-torus, ie. $D_k^{\mathbf{Z}^n} = \mathbf{G}_{m,k}^n$. Then there exists a canonical, $\mathbf{R}$ and $T(k)$-invariant isomorphism

$$\mathsf{Hom}_{\mathsf{Gps}}(D_k^{\mathbf{R}}, T) \cong X_*(T) \otimes \mathbf{R} \cong \mathbf{R}^n.$$

Proof. The idea of the proof is as follows: we imediately reduce to the case of $n = 1$ and we note that any homomorphism of groups $x\colon \mathbf{Z} \to \mathbf{R}$ singles out a morphism

$$\mathsf{Spec}\,k[T^{\mathbf{R}}] = D_k^{\mathbf{R}} \to D_k^{\mathbf{Z}} = \mathsf{Spec}\,k[T^{\pm}]$$

given by the functoriality of D. (In coordinates, we identify the right hand side as $\mathbf{Z}[T^{\pm}]$ and the morphism is given by $T \mapsto T^x$ on the left.)

Crucially now, we use the fact that this is a homomorphism of algebraic groups, hence of Hopf algebras, to see that any such homomorphism is of this form. (Hint: write the image of T as $\sum_x a_x T^x$. Use the fact it preseves multiplication to see that only one a_x is non-zero. The fact that it preserves inversion implies that this $a_x = 1$.) The proof now follows. □

Fix a split torus $G = \mathbf{G}_{m,K}^n$ and its canonical integral model $\mathcal{G} = \mathbf{G}_{m,A}$.

Now, we construct a map $\mathsf{Hom}_{\mathsf{Gps}}(D_K^{\mathbf{R}}, G) \to N^{\otimes}(G)$. Given such a map

$$\chi\colon D_K^{\mathbf{R}} \to G$$

we have now a unique extension to a homomorphism $D_A^{\mathbf{R}} \to \mathcal{G}$. This determines a norm

$$\alpha_\chi\colon \mathsf{Rep}^\circ(\mathcal{G}) \to \mathsf{Norm}^\circ(\mathcal{G})$$

given as follows.

Fix a $\mathcal{G}$-representation L over A. Then this determines a representation $D_K^{\mathbf{R}} \to \mathcal{G} \to \mathsf{GL}(L)$, and hence a decomposition into eigenspaces

$$V = \oplus_{x \in \mathbf{R}_{>0}} V_x.$$

This determines therefore a norm by the results of last sections. This assembles into a normed fiber functor (ie. is functorial, exact and endowed with a canonical symmetric monoidal structure).

Theorem 1. *The construction above determines a bijection*

$$X_*(T) \otimes \mathbf{R} \cong \mathsf{Hom}(D_K^{\mathbf{R}}, T) \xrightarrow{\sim} N^{\otimes}(T).$$

equivariant for both $G(K)$ and $\mathbf{R}$ actions.

We break the proof into three parts.

Proof (equivariance). How does $(K^\times)^n$ acts on both sides? On the left we see that each such tuple determines a morphism $T \to T$ and hence we get an action. On the right, we write for each normed vector space $\alpha(V)$ a decomposed lattice $L(V) = \oplus_x L(V)_x$ determining its norm, and we see that we must re-scale the norm acording to the tuple in question.

Staring at this long enough one sees that the construction above is equivariant essentially from its definition. □

Proof (injectivity). Since we have fixed a lattice in our construction (namely the canonical one associated to our base point $\mathcal{G}$) the norm is completely determined by the grading. □

Proof (surjectivity). By the "Main Theorem" of the paper, every norm is splittable. (More about this, and proof, on next talk). This gives us in particular another integral model $\widetilde{\mathcal{G}}$ of $G = \mathcal{G}_K$ given by the fiber functor

$$\lambda\colon \mathsf{Rep}^{\circ}\mathcal{G} \to \mathsf{Mod}_A$$

of our fiber functor ω together with a map $\chi\colon D_A^{\mathbf{R}} \to \widetilde{\mathcal{G}}$ splitting the norm globally.

Now $\mathcal{G}$ and $\widetilde{\mathcal{G}}$ are isomorphic fpqc-locally on A, and since we are working with smooth models, they are also isomorphic étale locally on A. But since any T-torsor on A is trivial (Satz90) they must be already isomorphic over A.

Since there was already a fixed isomorphism $\mathcal{G}_K \cong \widetilde{\mathcal{G}}_K$, we get an element $g \in G(K)$ and a computation shows that $g\chi$ splits $\mathcal{G}$ already. In other words, $g\theta_\chi$ is our normed fiber functor. By equivariance, the map above is surjective. $\square$