Algebraic Geometry II
SS 2026

Prof. Dr. Ulrich Görtz
Dr. Andreas Pieper

Problem sheet 2

Due date: May 5th, 2026.

Problem 4 (Closed subschemes of $\mathbb{P}^n$)

Let R be a ring, and let Z be a closed subscheme of $\mathbb{P}^n_R$. Show that there exists a homogeneous ideal I of $R[X_0, \dots, X_n]$ such that $Z = \mathcal{V}_+(I)$.

Hint: Define $I = (f \in R[X_0, \cdots, X_n]$ homogeneous : $\forall i, \frac{f}{X_i^{\deg(f)}} \in \mathfrak{a}_i)$, where $\mathfrak{a}_i \subset R[\frac{X_0}{X_i}, \cdots, \frac{X_n}{X_i}]$ is the ideal defining the closed subscheme $Z \cap D_+(X_i)$ of $D_+(X_i)$.

Problem 5 (Segre embedding) Let R be a ring. Show that there is a closed immersion

$$\mathbb{P}^n_R \times_{\operatorname{Spec} R} \mathbb{P}^m_R \hookrightarrow \mathbb{P}^{(n+1)(m+1)-1}_R$$

which on k-valued points is given by

$$((x_0 : \dots, x_n), (y_0 : \dots, y_m)) \mapsto (x_0 y_0 : \dots : x_i y_j : \dots : x_n y_m)$$

for any R-algebra k which is a field.

Problem 6 (Matrix rank)

Let k be an algebraically closed field, let $n, m \geqslant 1$ be integers, and let $V = \mathrm{Mat}_{n,m}(k)$ be the k-vector space of n by m matrices. Let $M_1, \cdots, M_{mn}$ be the standard basis of V. Let $P_{n,m} := \mathbb{P}^{mn-1}_k$, and identify the points of $P_{m,n}(k)$ with the elements of $(V \setminus \{0\})/k^\times$ by sending $(a_1 : \cdots : a_{mn})$ to the class of $\sum_i a_i M_i$. Show that the set of matrices of rank $\leqslant 1$ is closed in $P_{n,m}$.

Hint: Consider the morphism $P_{n,1} \times_{\operatorname{Spec} k} P_{1,m} \to P_{n,m}$ given by matrix multiplication.

Remark/Update: The same argument for rank $d > 1$ does not work, unlike what was suggested in the original version of the problem, because in that case the product of two non-zero matrices might be the zero matrix. Of course, in either case it is not difficult to show directly that the set in question is closed, because it is the common vanishing set of all $(d+1) \times (d+1)$-minors. (Using other methods it is possible to view this subset as the image of (the k-valued points of) a projective k-scheme in a natural way.)