Algebraic Geometry II Prof. Dr. Ulrich Görtz
SS 2026 Dr. Andreas Pieper

Problem sheet 3

Due date: May 12th, 2026.

Problem 7 (Hom sheaves) Let X be a ringed space and let $\mathscr{F}$ be an $\mathscr{O}_X$-module of finite presentation, i.e., every point $x \in X$ has an open neighborhood U such that there exist $m, n \geqslant 0$ and an exact sequence $\mathscr{O}_U^m \to \mathscr{O}_U^n \to \mathscr{F} \to 0$ of $\mathscr{O}_U$-modules. Show that for all $x \in X$ and for each $\mathscr{O}_X$-module $\mathscr{G}$, the canonical homomorphism of $\mathscr{O}_{X,x}$-modules $\mathscr{H}om_{\mathscr{O}_X}(\mathscr{F}, \mathscr{G})_x \to \mathrm{Hom}_{\mathscr{O}_{X,x}}(\mathscr{F}_x, \mathscr{G}_x)$ is bijective.

Hint. Use the exactness properties of passing to stalks, of the Hom functor and $\mathscr{H}om$ sheaf and the five lemma to reduce the general case to the case of a free $\mathscr{O}_X$-module.

Problem 8 (Blow-up)

Let k be a field. We consider the affine plane $\mathbb{A}^2_k$ with coordinates S, T, i.e., $\mathbb{A}^2_k = \operatorname{Spec} A$ with $A = k[S, T]$, and the projective line $\mathbb{P}^1_k$ with homogeneous coordinates X, Y over k.

Let $Z = V_+(SX - TY)$ be the closed subscheme of $\mathbb{A}^2_k \times_{\operatorname{Spec} k} \mathbb{P}^1_k = \mathbb{P}^1_A$ defined by the linear homogeneous polynomial $SX - TY \in A[X, Y]$. Let $f \colon Z \to \mathbb{A}^2_k$ be the restriction of the projection $\mathbb{P}^1_A \to \mathbb{A}^2_k$ to Z. Let $U = \mathbb{A}^2_k \smallsetminus \{0\}$, an open subscheme of $\mathbb{A}^2_k$.

1) Prove that the restriction $f^{-1}(U) \to U$ of f is an isomorphism.

2) Show that the fiber of f over $0 \in \mathbb{A}^2_k$ is isomorphic to $\mathbb{P}^1_k$.

Problem 9 (Finite morphisms) Recall that a ring homomorphism $A \to B$ is called *finite* if B is finitely generated as an A-module.

Definition 0.1. Let $f : X \to Y$ be a morphism of schemes. We say that f is *finite* if for any affine open $\operatorname{Spec}(A) = V \subseteq Y$ the inverse image $f^{-1}(V)$ is affine, say $f^{-1}(V) \cong \operatorname{Spec}(B)$ and the corresponding ring homomorphism $A \to B$ is finite.

1) Let $f : X \to Y$ be a morphism of schemes. Show that the following are equivalent:

   i) f is finite.

   ii) There exists an affine open covering $Y = \bigcup V_i$ such that each $f^{-1}(V_i)$ is affine and the ring homomorphism $\mathscr{O}_Y(V_i) \to \mathscr{O}_X(f^{-1}(V_i))$ is finite.

   Hint: You may use without proof that ii) implies that for every affine open $V \subseteq Y$, the inverse image $f^{-1}(V)$ is affine.

2) Show that finite morphisms are proper.
Hint: Use *going up*, see `https://math.ug/a2-ss23/sec-going-up.html`, specifically Korollar 4.18 (in German); or [Atiyah-MacDonald, Introduction to Commutative Algebra] Ch. 5, Exercise 1; or `https://stacks.math.columbia.edu/tag/00HU`.