Problem sheet 10

Due date: June 30th, 2026.

Problem 28 (Continuation of Problem 26) Let k be an algebraically closed field and (E, O_E) be an elliptic curve over k, i.e., a normal proper curve of genus 1 with a fixed point.

i) Show that the invertible sheaf $\mathscr{O}_E(3[O_E])$ induces a morphism $\iota : E \to \mathbb{P}^2_k$. Denote $C := \mathrm{im}(\iota)$, a closed integral subscheme of $\mathbb{P}^2_k$.

ii) Show that one can choose coordinates X, Y, Z on $\mathbb{P}^2_k$ so that the image of ι is contained in the homogenization of the Weierstraß equation found in Problem 26, i.e.,

$$C \subseteq V_+(X^2Z + a_1XYZ + a_3YZ^2 - (X^3 + a_2X^2Z + a_4XZ^2 + a_6Z^3))$$

for some $a_1, a_2, a_3, a_4, a_6 \in k$.

iii) Show that the right hand side of the inclusion in ii) is integral and deduce that the inclusion is an equality.

iv) Show that the map ι induces an isomorphism on fields of rational functions $K(C) \to K(E)$.
Hint: Consider the elements $x, y \in K(E)$ of Problem 28 and the subfields $k(x), k(y)$ generated by them. Study the corresponding morphisms $E \to \mathbb{P}^1_k$ and show that $[K(E) : k(x)] = 2$, $[K(E) : k(y)] = 3$. Furthermore, show that $x = \frac{X}{Z}, y = \frac{Y}{Z} \in K(C) \subseteq K(E)$.

Remark. One can show that C is a connected normal (and, of course, proper) curve. Therefore the above results imply that ι induces an isomorphism $E \cong C$ of E with a curve defined by a Weierstraß equation.

Problem 29 Let k be a field and X be a projective scheme over k. Let $\mathcal{L}$ be an invertible sheaf on X and $s_0, \dots s_n$ be a basis of its space of global sections. Assume that $\mathcal{L}$ with the s_i induces a morphism $\iota : X \to \mathbb{P}^n_k$ and assume in addition that ι is a closed immersion. Show that for each $i = 0, \dots, n$ the open subscheme $U = X \setminus V(s_i)$ is affine.
Hint: Prove that U is isomorphic to the preimage of a standard affine.

Problem 30 (Automorphisms of $\mathbb{P}^n$) Let k be a field and $n \geqslant 1$. Show that for any automorphism $\varphi : \mathbb{P}^n_k \to \mathbb{P}^n_k$ there exists an invertible matrix $A \in \mathrm{GL}_{n+1}(k)$ such that $\varphi = f_A$ where f_A is the map constructed in AG 1, Problem 38 (see the next page).

Algebraic Geometry 1, Problem 37

Let R be a ring, $n \geqslant 1$. Let $B = R[X_0, \dots, X_n]$. Let $Z = V(X_0, \dots, X_n) \subseteq \mathbb{A}^{n+1}_R = \operatorname{Spec} B$, and let $U = \mathbb{A}^{n+1}_R \smallsetminus Z$, an open subscheme of $\mathbb{A}^{n+1}_R$.

Show that there is a "natural" morphism $p\colon U \to \mathbb{P}^n_R$ of R-schemes such that for every field k the induced map $U(k) \to \mathbb{P}^n_R(k)$ is given by $(x_0, \dots, x_n) \mapsto (x_0\colon \dots : x_n)$.

Hint. Let $\mathbb{P}^n_R = \bigcup_{i=0}^n U_i$ be the standard affine cover. Define morphisms $D(X_i) \to U_i$ and construct p by gluing of morphisms, applied to the compositions $D(X_i) \to U_i \to \mathbb{P}^n_R$.

Algebraic Geometry 1, Problem 38

We continue to work in the setting of Problem 37. Let $A = (a_{ij})_{i,j} \in GL_{n+1}(R)$ be an invertible $(n+1) \times (n+1)$-matrix with entries in R.

(1) The ring isomorphism

$$B \to B, \quad X_i \mapsto \sum_j a_{ij} X_j,$$

induces an isomorphism $\mathbb{A}^{n+1}_R \to \mathbb{A}^{n+1}_R$ of R-schemes.

Show that A restricts to an automorphism $\widetilde{f}_A$ of U.

(2) Show that there exists a unique automorphism f_A of $\mathbb{P}^n_R$ which fits into a commutative diagram

$$\begin{array}{ccc} U & \xrightarrow{\widetilde{f}_A} & U \\ \downarrow & & \downarrow \\ \mathbb{P}^n_R & \xrightarrow{f_A} & \mathbb{P}^n_R. \end{array}$$

In this way we obtain a group homomorphism from $GL_{n+1}(R)$ into the group $\operatorname{Aut}_R(\mathbb{P}^n_R)$ of automorphisms of the R-scheme $\mathbb{P}^n_R$.

(3) Now let k be a field, $n = 1$. Let $\mathbb{P}^1_k = U_0 \cup U_1$ be the standard affine open cover. We have $\mathbb{P}^1(k) = U_0(k) \cup \{(0:1)\} = k \cup \{\infty\}$. Let $x, y, z \in \mathbb{P}^1(k)$ be distinct points. Show that there exists a unique automorphism f of $\mathbb{P}^1_k$ of the form f_A such that

$$f(0) = x, \quad f(1) = y, \quad f(\infty) = z.$$