Algebraic Geometry I Prof. Dr. Ulrich Görtz
WS 2025/26 Dr. Andreas Pieper

Problem sheet 11

Due date: Jan. 21, 2025.

Problem 35 For a ring R, consider $X = \mathbb{P}^n_R$. Show that $\mathcal{O}_X(X) \cong R$.

Problem 36 (Veronese map) Let k be a field. Show that there exists a morphism $\varphi : \mathbb{P}^2_k \to \mathbb{P}^5_k$ which induces the map

$\varphi(k)\colon \mathbb{P}^2_k(k) \to \mathbb{P}^5_k(k)$, $(x_0 : x_1 : x_2) \mapsto (x_0^2 : x_0x_1 : x_0x_2 : x_1^2 : x_1x_2 : x_2^2)$

on k-valued points.

Problem 37

Let R be a ring, $n \geqslant 1$. Let $B = R[X_0, \dots, X_n]$. Let $Z = V(X_0, \dots, X_n) \subseteq \mathbb{A}^{n+1}_R = \operatorname{Spec} B$, and let $U = \mathbb{A}^{n+1}_R \smallsetminus Z$, an open subscheme of $\mathbb{A}^{n+1}_R$.

Show that there is a “natural” morphism $p\colon U \to \mathbb{P}^n_R$ of R-schemes such that for every field k the induced map $U(k) \to \mathbb{P}^n_R(k)$ is given by $(x_0, \dots, x_n) \mapsto (x_0\colon \cdots : x_n)$.

Hint. Let $\mathbb{P}^n_R = \bigcup_{i=0}^n U_i$ be the standard affine cover. Define morphisms $D(X_i) \to U_i$ and construct p by gluing of morphisms, applied to the compositions $D(X_i) \to U_i \to \mathbb{P}^n_R$.

Problem 38

We continue to work in the setting of Problem 37. Let $A = (a_{ij})_{i,j} \in GL_{n+1}(R)$ be an invertible $(n+1) \times (n+1)$-matrix with entries in R.

(1) The ring isomorphism

$$B \to B, \quad X_i \mapsto \sum_j a_{ij} X_j,$$

induces an isomorphism $\mathbb{A}_R^{n+1} \to \mathbb{A}_R^{n+1}$ of R-schemes.

Show that A restricts to an automorphism $\widetilde{f}_A$ of U.

(2) Show that there exists a unique automorphism f_A of $\mathbb{P}_R^n$ which fits into a commutative diagram

$$\begin{array}{ccc} U & \xrightarrow{\widetilde{f}_A} & U \\ \downarrow & & \downarrow \\ \mathbb{P}_R^n & \xrightarrow{f_A} & \mathbb{P}_R^n. \end{array}$$

In this way we obtain a group homomorphism from $GL_{n+1}(R)$ into the group $\mathrm{Aut}_R(\mathbb{P}_R^n)$ of automorphisms of the R-scheme $\mathbb{P}_R^n$.

(3) Now let k be a field, $n = 1$. Let $\mathbb{P}_k^1 = U_0 \cup U_1$ be the standard affine open cover. We have $\mathbb{P}^1(k) = U_0(k) \cup \{(0 : 1)\} = k \cup \{\infty\}$. Let $x, y, z \in \mathbb{P}^1(k)$ be distinct points. Show that there exists a unique automorphism f of $\mathbb{P}_k^1$ of the form f_A such that

$$f(0) = x, \quad f(1) = y, \quad f(\infty) = z.$$