Algebraic Geometry I Prof. Dr. Ulrich Görtz
WS 2025/26 Dr. Andreas Pieper

Problem sheet 12

Due date: Jan. 28, 2025.

Problem 39 Let k be an algebraically closed field with $\mathrm{char}(k) \neq 2$. Consider $\mathbb{P}_k^2$ with coordinates x_0, x_1, x_2. A *projective conic* is a scheme of the form $\mathcal{V}_+(f)$ where $f \in k[X_0, x_1, x_2]$ is a homogeneous polynomial of degree 2.

a) Show that every homogeneous polynomial f of degree 2 may be expressed as xAx^t, where $A \in \mathrm{Mat}_{3,3}(k)$ is a non-zero symmetric matrix and x is the row vector (x_0, x_1, x_2). Correspondingly we write

$$Q_A := \mathcal{V}_+(xAx^t) \subset \mathbb{P}_k^2.$$

b) Show that there is a map $i : \mathbb{P}_k^1 \to \mathbb{P}_k^2$ wich on k-valued points is given by $(t_0 : t_1) \to (t_0^2 : t_0t_1 : t_1^2)$. Show that i induces an isomorphism $\mathbb{P}_k^1 \cong \mathcal{V}_+(x_0x_2 - x_1^2) \subset \mathbb{P}_k^2$.

c) Recall from linear algebra (or convince yourself, but you do not have to write this out) that for any invertible symmetric matrices A, B there exists an invertible matrix S such that $SAS^t = B$.

d) Deduce that any projective conic Q_A with A invertible is isomorphic to $\mathbb{P}_k^1$.

e) Consider the standard affine open $\mathbb{A}_k^2 \cong U_0 \subset \mathbb{P}_k^2$. Show that for any invertible symmetric matrix A, the intersection $\mathbb{A}_k^2 \cap Q_A$ is an affine conic (Problem 34). Prove that $\mathbb{A}_k^2 \cap Q_A$ is isomorphic to either $\mathbb{A}_k^1$ or $\mathbb{A}_k^1 \smallsetminus \{0\}$.

Problem 40

Let k be a field. Let X be the scheme obtained by gluing two copies of $\mathbb{A}_k^1 = \operatorname{Spec} k[T]$ along the open subset $U = D(T)$, with respect to the identity map $U \to U$ ("the affine line with the origin doubled").

a) Compute $\mathcal{O}_X(X)$.

b) Deduce that X is not an affine scheme.

c) Show that there exist two distinct morphisms $j_1, j_2 : \mathbb{A}^1_k \to X$ such that $j_{1|D(T)} = j_{2|D(T)}$.

Problem 41 Let S, X_1, X_2, Y be schemes. Suppose we are given morphisms $X_1 \to X_2, X_2 \to S, Y \to S$. Show that

$$X_1 \times_S Y \cong X_1 \times_{X_2} (X_2 \times_S Y).$$

Hint: Here is a visualization:

$$\begin{array}{ccccc} X_1 \times_S Y & \longrightarrow & X_2 \times_S Y & \longrightarrow & Y \\ \downarrow & & \downarrow & & \downarrow \\ X_1 & \longrightarrow & X_2 & \longrightarrow & S \end{array}$$

The result holds in an arbitrary category where the fiber products exist (and correspondingly, you only need to use the universal property of the fiber product to prove the statement).