Algebraic Geometry I Prof. Dr. Ulrich Görtz
WS 2025/26 Dr. Andreas Pieper

Problem sheet 13

Due date: Feb. 4, 2026.

Problem 42 Let k be a field. Consider the projective quadric

$$X := \mathcal{V}_+(x_0x_3 - x_1x_2) \subset \mathbb{P}^3_k.$$

a) Show that there exists a map $\varphi : X \to \mathbb{P}^1_k \times \mathbb{P}^1_k$ which on k-points is given by $(a_0 : a_1 : a_2 : a_3) \mapsto ((a_0 : a_1), (a_0 : a_2))$ for all $(a_0 : a_1 : a_2 : a_3) \in X(k)$ such that $a_0 \neq 0$.

b) Show that φ is an isomorphism.

Problem 43 Let k be an algebraically closed field. Consider the morphism $\mathbb{A}^1_k \to \mathbb{A}^1_k$ corresponding to the ring homomorphism $k[t] \to k[t]$, $t \mapsto t^2$. Describe the scheme theoretic fibers $f^{-1}(x)$ for all $x \in \mathbb{A}^1_k(k)$. Which fibers are reduced?

Problem 44 Let $\varphi : X \to S$, $S' \to S$ be maps of schemes. The *base-change* of φ is defined to be the map $\varphi' : X \times_S S' \to S'$.

$$\begin{array}{ccc} X \times_S S' & \xrightarrow{\varphi'} & S' \\ \downarrow & & \downarrow \\ X & \xrightarrow{\varphi} & S \end{array}$$

a) Let K be a field, and let L_1, L_2 be extension fields of K. Show that there exists a field L together with K-homomorphisms $L_1 \to L$, $L_2 \to L$. (*Hint.* To construct L, you could start with the tensor product $L_1 \otimes_K L_2$ (however note that this will not in general be a field).)

b) Show that the surjectivity of φ implies the surjectivity of φ'.

c) Give an example of a situation where φ is injective, but φ' is not.

(We call a morphism of schemes *surjective* (or *injective*, *bijective*), if the underlying continuous map has the respective property. Note that every isomorphisms of schemes is bijective; but the converse is not true.)