Algebraic Geometry I Prof. Dr. Ulrich Görtz
WS 2025/26 Dr. Andreas Pieper

Problem sheet 8

Due date: Dec. 16, 2025.

Problem 26 Let X be a topological space, $U \subseteq X$ open, and denote by $j\colon U \to X$ the inclusion map. Let $\mathscr{F}$ be a sheaf of abelian groups on U. Denote by $j_!(\mathscr{F})$ the sheaf associated with the following presheaf on X:

$$V \mapsto \begin{cases} \mathscr{F}(V) & \text{if } V \subseteq U, \\ 0 & \text{otherwise,} \end{cases} \qquad \text{for } V \subseteq X \text{ open}$$

Compute the stalks of $j_!(\mathscr{F})$ and the restriction $j_!(\mathscr{F})_{|U}$. It is possible to define $j_!$ on sheaf morphisms, so that $j_!$ is a functor. Show that j^{-1} is right adjoint to $j_!$.

Remark. Recall that $j^{-1}\mathscr{G} = \mathscr{G}_{|U}$. You do not have to write out the proof that the bijections $\operatorname{Hom}(j_!\mathscr{F}, \mathscr{G}) \xrightarrow{\cong} \operatorname{Hom}(\mathscr{F}, j^{-1}\mathscr{G})$ are functorial in $\mathscr{F}$ and $\mathscr{G}$.

Problem 27 Give an example of affine schemes X, Y and a morphism $X \to Y$ of ringed spaces which is not a morphism of locally ringed spaces.
Hint: Consider a DVR R and its field of fractions K. Construct a morphism of ringed spaces $\operatorname{Spec}(K) \to \operatorname{Spec}(R)$ whose image is the closed point.

Problem 28 (Gluing topological spaces) Let U_1, U_2, U_3 be topological spaces. Suppose we are given:

- $U_{ij} \subseteq U_i$, $i, j \in \{1, 2, 3\}$ open,
- isomorphisms $\varphi_{ji} : U_{ij} \xrightarrow{\cong} U_{ji}$.

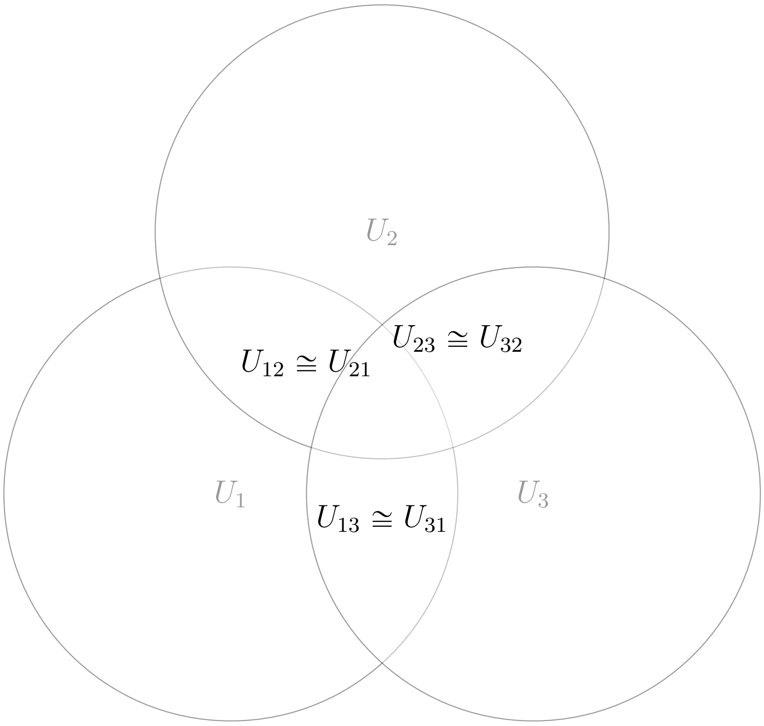
such that

(a) $U_{ii} = U_i$ and

(b) the cocycle condition $\varphi_{kj} \circ \varphi_{ji} = \varphi_{ki}$ holds on $U_{ij} \cap U_{ik}$ for all $i, j, k = 1, 2, 3$.

Show that there exists a topological space X together with open embeddings $\psi_i : U_i \to X$ such that

- $\psi_j \circ \varphi_{ji} = \psi_i$ on U_{ij} for all $i, j = 1, 2, 3$,
- $X = \bigcup_{i=1}^{3} \varphi_i(U_i)$
- $\psi_i(U_i) \cap \psi_j(U_j) = \psi_i(U_{ij}) = \psi_j(U_{ji})$ for all $i, j = 1, 2, 3$.



Bonus exercise: Generalize to arbitrarily many U_i. Show that (X, ψ_i) is unique up to unique isomorphism.